7. novembar

Brojni nizovi

1. Odrediti opšti član sledećih nizova

a) $-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}, \dots$

$$a_n = \frac{(-1)^n}{n^2}$$

c) $\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \dots$

$$a_n = \frac{n+2}{n+1}$$

b) $\frac{1}{2}, -1, \frac{3}{2}, -2, \frac{5}{2}, -3 \dots$

$$a_n = (-1)^{n-1} \cdot \frac{n}{2}$$

d) $\frac{1}{1\cdot 2}, \frac{1}{2\cdot 3}, \frac{1}{3\cdot 4}, \dots$

$$a_n = \frac{1}{n(n+1)}$$

2. Odrediti 3. član niza čiji je opšti član $a_n = 1 + \frac{1}{2} + \dots + \frac{1}{2^{n+1}}$

$$a_1 = 1 + \frac{1}{2} + \frac{1}{2^2}$$

$$a_2 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}$$

$$a_3 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$$

3. Dat je niz (a_n) čiji je opšti član $a_n = \frac{n(n+1)}{2}$. Napisati a_1, a_2, a_3, a_{25}, a_{100}, a_{n-3}.

$$a_1 = \frac{1\cdot 2}{2} = 1 \qquad a_{25} = \frac{25 \cdot \cancel{26}^{13}}{\cancel{2}} = 325 \qquad a_{n-3} = \frac{(n-3)(n-2)}{2}$$

$$a_2 = 3 \qquad a_{100} = \frac{100 \cdot 101}{2} = 5050$$

$$a_3 = 6$$

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Aritmetički niz: $a_1, a_1 + d, a_1 + 2d, \dots$

$$a_1$$

$$a_2 = a_1 + d$$

$$a_3 = a_1 + 2d$$

$$\vdots$$

$$a_n = a_1 + (n-1)d$$

$$S_n = a_1 + a_2 + \dots + a_n$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_n = \frac{n}{2}(2a_1 + (n-1)d)$$

4. Naći sumu prvih 20 članova niza $2, 5, 8, 11, \dots$

$$a_1 = 2, \quad d = 3, \quad S_{20} = ?$$

$$S_{20} = \frac{20}{2} \cdot (2 \cdot 2 + 19 \cdot 3) = 10 \cdot (4 + 57) = 610$$

5. Naći sumu prvih n prirodnih brojeva.

$$1, 2, 3, \dots$$

$$a_1 = 1, \quad d = 1, \quad S_n = ?$$

$$S_n = \frac{n}{2}(2 \cdot 1 + (n-1) \cdot 1) = \frac{n(n+1)}{2}$$

Geometrijski niz

$$a_1, a_1 \cdot q, a_1 \cdot q^2, \dots$$

a_1

$a_2 = a_1 \cdot q$

$a_3 = a_1 \cdot q^2$

$\vdots$

$a_n = a_1 \cdot q^{n-1}$

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

$$S_n = a_1 \frac{1-q^n}{1-q} \quad \left(\frac{q^n - 1}{q-1} \right)$$

6. Naći sumu $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots (-1)^n \cdot \frac{1}{2^{n+1}}$

$$a_1 = 1, \quad q = -\frac{1}{2}, \quad S_{n+2} = ?$$

$$S_{n+2} = 1 \cdot \frac{1 - \left(-\frac{1}{2}\right)^{n+2}}{1 - \left(-\frac{1}{2}\right)} = \frac{2}{3}\left(1 - \left(-\frac{1}{2}\right)^{n+2}\right)$$

1) $\lim\limits_{n \to \infty} \frac{1}{n} = 0$

2) $\lim\limits_{n \to \infty} \sqrt[n]{n} = 1$

3) $\lim\limits_{n \to \infty} \sqrt[n]{a} = 1, \quad a > 0$

4) $\lim\limits_{n \to \infty} \frac{n^k}{a^n} = 0, \quad a > 1$

5) $\lim\limits_{n \to \infty} \frac{a^n}{n!} = 0$

6) $\lim\limits_{n \to \infty} q^n = \begin{cases} 0, & |q| < 1 \\ 1, & q = 1 \\ +\infty, & q > 1 \end{cases}$

$$1.\quad \lim_{n\to\infty} \frac{(2n+1)(n+1)n}{n^3+3n^2+3n+1} = \lim_{n\to\infty} \frac{2n^3+3n^2+n}{n^3+3n^2+3n+1} = \lim_{n\to\infty} \frac{n^3\left(2+\frac{3}{n}+\frac{1}{n^2}\right)}{n^3\left(1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}\right)} = \frac{2}{1}$$

14. novembar

$$1.\quad \lim_{n\to\infty} \frac{1+2+\ldots+n}{n^2} = \lim_{n\to\infty} \frac{\frac{n}{2}(1+n)}{n^2} = \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(1+\frac{1}{n}\right)}{2\cancel{n^2}} = \frac{1}{2}$$

$$2.\quad \lim_{n\to\infty} \frac{1\cdot 2+2\cdot 3+\ldots+n(n+1)}{n^3} = \lim_{n\to\infty} \frac{1\cdot 2+2\cdot 3+\ldots+n^2+n}{n^3} = \lim_{n\to\infty} \frac{1^2+1+2^2+2+\ldots+n^2+n}{n^3}$$

$n=1 \quad 1\cdot 2 = 1^2+1$

$n=2 \quad 2\cdot 3 = 2^2+2$

$\vdots$

$n(n+1) = n^2+n$

$$= \lim_{n\to\infty} \frac{1+2+\ldots+n+1^2+2^2+\ldots+n^2}{n^3} = \lim_{n\to\infty} \frac{\frac{n(n+1)}{2}+\frac{n(n+1)(2n+1)}{6}}{n^3} =$$

$$= \lim_{n\to\infty} \frac{3n(n+1)+n(n+1)(2n+1)}{6n^3} = \lim_{n\to\infty} \frac{n(n+1)(3+2n+1)}{6n^3} = \lim_{n\to\infty} \frac{2n(n+1)(n+2)}{6n^3} =$$

$$= \frac{2}{6}\lim_{n\to\infty} \frac{n^3+3n^2+2n}{n^3} = \frac{1}{3}$$

$$3.\quad \lim_{n\to\infty}\left(n-2-\sqrt{n^2-n+1}\right) = \lim_{n\to\infty}\left(n-2-\sqrt{n^2-n+1}\right)\cdot\frac{n-2+\sqrt{n^2-n+1}}{n-2+\sqrt{n^2-n+1}} =$$

$$= \lim_{n\to\infty} \frac{(n-2)^2-(n^2-n+1)}{n-2+\sqrt{n^2\left(1-\frac{1}{n}+\frac{1}{n^2}\right)}} = \lim_{n\to\infty} \frac{-3n+3}{n-2+n\sqrt{1-\frac{1}{n}+\frac{1}{n^2}}} = \lim_{n\to\infty} \frac{\cancel{n}\left(-3+\frac{3}{n}\right)}{\cancel{n}\left(1-\frac{2}{n}+\sqrt{1-\frac{1}{n}+\frac{1}{n^2}}\right)}$$

$$= \frac{-3}{2}$$

$$4.\quad \lim_{n\to\infty} \sqrt[3]{n^2}\left(\sqrt[3]{n-1}-\sqrt[3]{n+1}\right) = \lim_{n\to\infty} \sqrt[3]{n^2}\left(\sqrt[3]{n-1}-\sqrt[3]{n+1}\right)\cdot\frac{\sqrt[3]{(n-1)^2}+\sqrt[3]{n-1}\cdot\sqrt[3]{n+1}+\sqrt[3]{(n+1)^2}}{\sqrt[3]{(n-1)^2}+\sqrt[3]{n-1}\cdot\sqrt[3]{n+1}+\sqrt[3]{(n+1)^2}}$$

$$= \lim_{n\to\infty} \sqrt[3]{n^2}\,\frac{n-1-(n+1)}{\sqrt[3]{(n-1)^2}+\sqrt[3]{(n-1)(n+1)}+\sqrt[3]{(n+1)^2}} = \lim_{n\to\infty} \sqrt[3]{n^2}\,\frac{-2}{\sqrt[3]{n^2\left(1-\frac{1}{n}\right)^2}+\sqrt[3]{n^2\left(1-\frac{1}{n^2}\right)}+\sqrt[3]{n^2\left(1+\ldots\right.}}$$

$$= \lim_{n\to\infty} \cancel{\sqrt[3]{n^2}} \frac{-2}{\cancel{\sqrt[3]{n^2}}\left(\sqrt[3]{(1-\frac{1}{n})^2} + \sqrt[3]{1-\frac{1}{n^2}} + \sqrt[3]{(1+\frac{1}{n})^2}\right)} = \frac{-2}{3}$$

$$\circledast \lim_{n\to\infty}\left(\sqrt{n^2+\sqrt{n}} - 2n + \sqrt{n^2+(-1)^n}\right) = \lim_{n\to\infty}\left(\sqrt{n^2+\sqrt{n}} - n - n + \sqrt{n^2+(-1)^n}\right) =$$

$$= \lim_{n\to\infty}\left(\sqrt{n^2+\sqrt{n}} - n\right) - \lim_{n\to\infty}\left(n - \sqrt{n^2+(-1)^n}\right) = \lim_{n\to\infty}\left(\sqrt{n^2+\sqrt{n}} - n\right)\cdot\frac{\sqrt{n^2+\sqrt{n}}+n}{\sqrt{n^2+\sqrt{n}}+n} -$$

$$- \lim_{n\to\infty}\left(n - \sqrt{n^2+(-1)^n}\right)\cdot\frac{n+\sqrt{n^2+(-1)^n}}{n+\sqrt{n^2+(-1)^n}} = \lim_{n\to\infty}\frac{\cancel{n^2}+\sqrt{n}-\cancel{n^2}}{\sqrt{n^2+\sqrt{n}}+n} - \lim_{n\to\infty}\frac{\cancel{n^2}-\cancel{n^2}-(-1)^n}{n+\sqrt{n^2+(-1)^n}} =$$

$$= \lim_{n\to\infty}\frac{\cancel{n}\cdot\frac{1}{\sqrt{n}}}{\cancel{n}\left(\sqrt{1+\frac{\sqrt{n}}{n^2}}+1\right)} + \lim_{n\to\infty}\frac{(-1)^n}{n\left(1+\sqrt{1+\frac{(-1)^n}{n^2}}\right)} = 0 + 0 = 0$$

$$6.\ \lim_{n\to\infty}\frac{(-2)^n+5^n}{(-2)^{n+1}+5^{n+1}} = \lim_{n\to\infty}\frac{5^n\left(\left(-\frac{2}{5}\right)^n+1\right)}{5^{n+1}\left(\left(-\frac{2}{5}\right)^{n+1}+1\right)} = \frac{1}{5}$$

$$\lim_{n\to\infty} q^n = 0$$

$$|q| < 1$$

$$7.\ \lim_{n\to\infty}\frac{1+2+2^2+\dots+2^n}{1+3+3^2+\dots+3^n} = \lim_{n\to\infty}\frac{1\cdot\frac{1-2^{n+1}}{1-2}}{1\cdot\frac{1-3^{n+1}}{1-3}} = \lim_{n\to\infty} 2\,\frac{1-2^{n+1}}{1-3^{n+1}} =$$

$$= 2\lim_{n\to\infty}\frac{2^{n+1}\left(\frac{1}{2^{n+1}}-1\right)}{3^{n+1}\left(\frac{1}{3^{n+1}}-1\right)} = 2\lim_{n\to\infty}\left(\frac{2}{3}\right)^{n+1}\cdot\frac{\frac{1}{2^{n+1}}-1}{\frac{1}{3^{n+1}}-1} = 2\cdot 0 = 0$$

$$8.\ \lim_{n\to\infty}\sqrt{2}\cdot\sqrt[4]{2}\cdot\sqrt[8]{2}\cdot\ldots\cdot\sqrt[2^n]{2} = \lim_{n\to\infty} 2^{\frac{1}{2}}\cdot 2^{\frac{1}{4}}\cdot 2^{\frac{1}{8}}\cdot\ldots\cdot 2^{\frac{1}{2^n}} =$$

$$= \lim_{n\to\infty} 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots+\frac{1}{2^n}} = \lim_{n\to\infty} 2^{\frac{1}{2}\cdot\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}} = \lim_{n\to\infty} 2^{1-\left(\frac{1}{2}\right)^n} = 2^1 = 2$$

9. Odrediti gr. vr. niza čiji je opšti član $a_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}$

$$a_1 = \frac{1}{1\cdot 2} = \frac{1}{2}, \quad a_2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3}$$

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$\frac{1}{n(n+1)} = \frac{A(n+1) + Bn}{n(n+1)} \Rightarrow A(n+1) + Bn = 1$$

$$(A+B)n + A = 1 \Rightarrow \left.\begin{array}{l} A+B=0 \\ A=1 \end{array}\right\} \Rightarrow \begin{array}{l} A=1 \\ B=-1 \end{array}$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$n=1, \quad \frac{1}{1\cdot 2} = \frac{1}{1} - \frac{1}{2}$$

$$n=2, \quad \frac{1}{2\cdot 3} = \frac{1}{2} - \frac{1}{3}$$

$$n=3, \quad \frac{1}{3\cdot 4} = \frac{1}{3} - \frac{1}{4}$$

$$a_n = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4} + \ldots + \frac{1}{n} - \frac{1}{n+1}$$

$$a_n = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 - \cancelto{0}{\frac{1}{n+1}}\right) = 1$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

10. $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^{-1}\right)^n = \lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^n\right)^{-1} = \lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right)^{-1} = e^{-1} = \frac{1}{e}$

11. $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n\cdot\frac{1}{2}} = e^{\frac{1}{2}} = \sqrt{e}$

12. $\lim\limits_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n = \lim\limits_{n\to\infty}\left(1+\frac{n+1}{n-1}-1\right)^n = \lim\limits_{n\to\infty}\left(1+\frac{2}{n-1}\right)^n = \lim\limits_{n\to\infty}\left(1+\frac{1}{\frac{n-1}{2}}\right)^{\frac{n-1}{2}\cdot\frac{2}{n-1}}$

$= \lim\limits_{n\to\infty}\left(\left(1+\frac{1}{\frac{n-1}{2}}\right)^{\frac{n-1}{2}}\right)^{\frac{2n}{n-1}} = e^2$

13. Primjenom teoreme o monotonom i ograničenom nizu dokazati da niz konvergira.

a) $a_n = \frac{1}{3+1} + \frac{1}{3^2+1} + \dots + \frac{1}{3^n+1}$

$a_{n+1} - a_n = \frac{1}{3+1} + \frac{1}{3^2+1} + \dots + \frac{1}{3^n+1} + \frac{1}{3^{n+1}+1} - \left(\frac{1}{3+1} + \frac{1}{3^2+1} + \dots + \frac{1}{3^n+1}\right) =$

$= \frac{1}{3^{n+1}+1} > 0$

$a_{n+1} - a_n > 0, \ \forall n \in \mathbb{N}$

$a_{n+1} > a_n, \ \forall n \in \mathbb{N} \Rightarrow (a_n)$ je monotono rastući

Kako je niz (a_n) rastući to je $a_1 \leq a_n, \ \forall n \in \mathbb{N}$

$\frac{1}{4} \leq a_n, \ \forall n \in \mathbb{N}$

$a_n = \frac{1}{3+1} + \frac{1}{3^2+1} + \dots + \frac{1}{3^n+1} \leq \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^n} = \frac{1}{3} \cdot \frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}} =$

$= \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{2}\left(1-\left(\frac{1}{3}\right)^n\right) = \frac{1}{2}\left(1-\left(\frac{1}{3}\right)^n\right)$

$a_n \leq \frac{1}{2}\left(1-\left(\frac{1}{3}\right)^n\right) < \frac{1}{2} \cdot 1$

$\frac{1}{4} < a_n < \frac{1}{2}, \ \forall n \in \mathbb{N} \Rightarrow (a_n)$ je ograničen

$\left.\begin{array}{l} (a_n) \text{ je monoton niz} \\ (a_n) \text{ je ograničen niz} \end{array}\right\} \Rightarrow$ niz (a_n) konvergira

b) $a_n = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdot\ldots\cdot\left(1-\frac{1}{2^{n+1}}\right)$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{\left(1-\frac{1}{2}\right)}\cancel{\left(1-\frac{1}{4}\right)}\cdot\ldots\cdot\cancel{\left(1-\frac{1}{2^{n+1}}\right)}\left(1-\frac{1}{2^{n+2}}\right)}{\cancel{\left(1-\frac{1}{2}\right)}\cancel{\left(1-\frac{1}{4}\right)}\cdot\ldots\cdot\cancel{\left(1-\frac{1}{2^{n+1}}\right)}} = 1 - \frac{1}{2^{n+2}} < 1$$

$$\frac{a_{n+1}}{a_n} < 1 \implies a_{n+1} < a_n \, , \; \forall n \in N$$

$a_n > 0$ $\qquad$ (a_n) monotono opadajući niz $\quad (1)$

Kako je niz (a_n) opadajući to važi $a_n \leq a_1 \, , \; \forall n \in N$

$$a_1 = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{2^2}\right) = \frac{1}{2}\cdot\frac{3}{4} = \frac{3}{8}$$

$0 < a_n \leq \frac{3}{8} \, , \; \forall n \in N \implies$ niz (a_n) je ograničen (2)

Iz (1) i (2) slijedi da je niz (a_n) konvergentan

c) $a_n = \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \ldots + \frac{1}{(3n+1)(3n+4)}$

$$\frac{1}{(3n+1)(3n+4)} = \frac{A}{3n+1} + \frac{B}{3n+4}$$

$A = \frac{1}{3}$

$B = -\frac{1}{3}$

$$\frac{1}{(3n+1)(3n+4)} = \frac{1}{3(3n+1)} - \frac{1}{3(3n+4)}$$

$$n = 1 \, , \quad \frac{1}{4\cdot 7} = \frac{1}{3\cdot 4} - \frac{1}{3\cdot 7} = \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7}\right)$$

$$n = 2 \, , \quad \frac{1}{7\cdot 10} = \frac{1}{3\cdot 7} - \frac{1}{3\cdot 10} = \frac{1}{3}\left(\frac{1}{7} - \frac{1}{10}\right)$$

$$a_n = \frac{1}{3}\left(\frac{1}{4} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{7}} - \cancel{\frac{1}{10}} + \ldots + \cancel{\frac{1}{3n+1}} - \frac{1}{3n+4}\right)$$

$$a_n = \frac{1}{3}\left(\frac{1}{4} - \frac{1}{3n+4}\right)$$

$0 < a_n < \frac{1}{3} \cdot \frac{1}{4}$

$0 < a_n < \frac{1}{12}, \quad \forall n \in N \Rightarrow (a_n)$ je ograničen (1)

$a_{n+1} - a_n = \frac{1}{(3(n+1)+1)\cdot(3(n+1)+4)} > 0, \quad \forall n \in N \Rightarrow (a_n)$ je monotono rastući

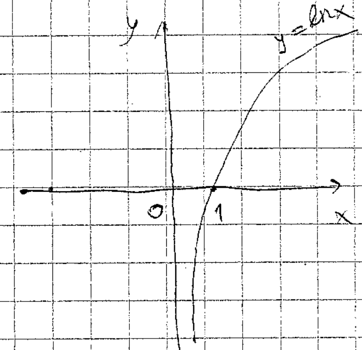
Iz (1) i (2) slijedi da (a_n) konvergira

d) $x_n = \frac{\ln(1+1)}{1\cdot 2} + \frac{\ln(1+\frac{1}{2})}{2\cdot 3} + \ldots + \frac{\ln(1+\frac{1}{n})}{n(n+1)}$

$x_{n+1} - x_n = \frac{\ln(1+\frac{1}{n+1})}{(n+1)(n+2)} > 0, \quad \forall n \in N$

$x_{n+1} > x_n, \quad \forall n \in N$

(x_n) je monotono rastući



$x_n > 0, \quad \forall n \in N$ može i $x_n > \frac{\ln 2}{2}$

$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$x_n \leq \frac{\ln 2}{1\cdot 2} + \frac{\ln 2}{2\cdot 3} + \ldots + \frac{\ln 2}{n(n+1)} = \ln 2\left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}\right) =$

$= \ln 2\left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \ldots + \frac{1}{n} - \frac{1}{n+1}\right) = \ln 2 \cdot \left(1 - \frac{1}{n+1}\right) < \ln 2 \cdot 1 = \ln 2,$

$\forall n \in N$

$0 < x_n < \ln 2, \quad \forall n \in N \Rightarrow (x_n)$ je ograničen (2)

Iz (1) i (2) slijedi da niz (x_n) konvergira.

14. Primjenom teoreme o uklještenju dokazati konvergenciju i naći graničnu vrijednost niza.

a) $$x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}$$

$$x_n \leq \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \dots + \frac{1}{\sqrt{n^2+1}} = \frac{n}{\sqrt{n^2+1}}$$

$$x_n \geq \frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+n}} + \dots + \frac{1}{\sqrt{n^2+n}} = \frac{n}{\sqrt{n^2+n}}$$

$$n \cdot \frac{1}{\sqrt{n^2+1}} \leq x_n \leq n \cdot \frac{1}{\sqrt{n^2+n}}$$

$$a_n = \frac{n}{\sqrt{n^2+n}}, \quad b_n = \frac{n}{\sqrt{n^2+1}}$$

1° $a_n \leq x_n \leq b_n$

2° $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\sqrt{n^2(1+\frac{1}{n})}} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\sqrt{1+\cancel{\frac{1}{n}}^{\,0}}} = 1$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\sqrt{1+\cancel{\frac{1}{n^2}}^{\,0}}} = 1$$

Iz 1°, 2° slijedi da je $\lim_{n\to\infty} x_n = 1$

$$\boxed{x_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}}$$

b) $$a_n = \frac{n+(-1)^1}{\sqrt[5]{n^{10}+1}} + \frac{n+(-1)^2}{\sqrt[5]{n^{10}+2}} + \dots + \frac{n+(-1)^n}{\sqrt[5]{n^{10}+n}}$$

$$a_n \leq \frac{n+1}{\sqrt[5]{n^{10}+1}} + \frac{n+1}{\sqrt[5]{n^{10}+1}} + \dots + \frac{n+1}{\sqrt[5]{n^{10}+1}} = n \cdot \frac{n+1}{\sqrt[5]{n^{10}+1}}$$

$$a_n \geq \frac{n-1}{\sqrt[5]{n^{10}+n}} + \frac{n-1}{\sqrt[5]{n^{10}+n}} + \dots + \frac{n-1}{\sqrt[5]{n^{10}+n}} = n \cdot \frac{n-1}{\sqrt[5]{n^{10}+n}}$$

$$\frac{n(n+1)}{\sqrt[5]{n^{10}+n}} \leq a_n \leq \frac{n(n+1)}{\sqrt[5]{n^{10}+1}}$$

$$x_n = \frac{n(n+1)}{\sqrt[5]{n^{10}+n}}, \quad y_n = \frac{n(n+1)}{\sqrt[5]{n^{10}+1}}$$

1° $x_n \leq a_n \leq y_n, \ \forall n \in \mathbb{N}$

2° $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{\cancel{n^2}(1-\cancel{\frac{1}{n}}^{0})}{\cancel{n^2}\sqrt[5]{1+\frac{1}{n^9}}} = 1$

$\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} \frac{\cancel{n^2}(1+\frac{1}{n})}{\cancel{n^2}\sqrt[5]{1+\frac{1}{n^{10}}}} = 1$

Iz 1° i 2° slijedi $\lim\limits_{n\to\infty} a_n = 1$

c) $X_n = \sum\limits_{k=1}^{n} \frac{5n}{3n^2-2k}$

$$X_n = \frac{5n}{3n^2-2} + \frac{5n}{3n^2-4} + \dots + \frac{5n}{3n^2-2n}$$

$$X_n \leq \frac{5n}{3n^2-2n} + \frac{5n}{3n^2-2n} + \dots + \frac{5n}{3n^2-2n}$$

$$X_n \leq n \cdot \frac{5n}{3n^2-2n}$$

$$X_n \geq \frac{5n}{3n^2-2} + \frac{5n}{3n^2-2} + \dots + \frac{5n}{3n^2-2} = n \frac{5n}{3n^2-2}$$

$$a_n = \frac{5n^2}{3n^2-2}, \quad b_n = \frac{5n^2}{3n^2-2n}$$

$$\left.\begin{array}{l} 1^\circ \ a_n \leq X_n \leq b_n \\ 2^\circ \ \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \frac{5}{3} \end{array}\right\} \Rightarrow \lim\limits_{n\to\infty} X_n = \frac{5}{3}$$

$n+1$ članova

d) $$\lim_{n\to\infty} \underbrace{\left(\sum_{k=0}^{n} \frac{2n+\sqrt{k}}{n^2-\sqrt{k}}\right)}_{a_n}$$

e) $$x_n = \sum_{k=1}^{n} \frac{(-1)^k + n^2}{\sqrt[3]{n^9+(-1)^k\cdot k} + \sin(k!)}$$

$$x_n \leq n\,\frac{n^2+1}{\sqrt[3]{n^9-n}-1}$$

$$x_n \geq n\,\frac{n^2-1}{\sqrt[3]{n^9+n}+1},$$

$$a_n = \frac{n(n^2-1)}{\sqrt[3]{n^9+n}+1}, \qquad b_n = \frac{n(n^2+1)}{\sqrt[3]{n^9-n}-1}$$

1° $a_n \leq x_n \leq b_n, \ \forall n \in \mathbb{N}$

2° $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\cancel{n^3}\left(1-\overset{\to 0}{\frac{1}{n^2}}\right)}{\cancel{n^3}\left(\sqrt[3]{1+\underset{\to 0}{\frac{1}{n^8}}}+\underset{\to 0}{\frac{1}{n^3}}\right)} = 1$$

$$\lim_{n\to\infty} b_n = \qquad = 1$$

Iz 1° i 2° slijedi da je $\lim_{n\to\infty} x_n = 1$

f) $$x_n = \sqrt[n]{4^n+3^n-\sin(n!)}$$

$$x_n = \sqrt[n]{4^n\left(1+\left(\frac{3}{4}\right)^n-\frac{\sin(n!)}{4^n}\right)}$$

$$x_n = 4\sqrt[n]{1+\left(\frac{3}{4}\right)^n-\frac{\sin(n!)}{4^n}}$$

$$x_n \leq 4\sqrt[n]{1+\frac{3}{4}+\frac{1}{4}} = 4\sqrt[n]{2}$$

$$x_n \geq 4\sqrt[n]{1+0-\frac{1}{4}} = 4\sqrt[n]{\frac{3}{4}}$$

$$4\sqrt[n]{\frac{3}{4}} \leq x_n \leq 4\sqrt[n]{2}$$

$$a_n = 4\sqrt[n]{\frac{3}{4}}, \quad b_n = 4\sqrt[n]{2}$$

$$\left.\begin{array}{l} 1^\circ \quad a_n \leq x_n \leq b_n, \quad \forall n \in \mathbb{N} \\ 2^\circ \quad \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 4\sqrt[n]{\frac{3}{4}} = 4 \cdot 1 = 4 \\ \quad\;\; \lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} 4\sqrt[n]{2} = 4 \cdot 1 = 4 \end{array}\right\} \Rightarrow \lim_{n\to\infty} x_n = 4$$

g) $a_n = \sqrt[3]{n+\sin n} - \sqrt[3]{n+\cos\frac{1}{n}}$

$$a_n \leq \sqrt[3]{n+1} - \sqrt[3]{n-1}$$

$$a_n \leq \sqrt[3]{n+1} - \sqrt[3]{n-1} = \left(\sqrt[3]{n+1} - \sqrt[3]{n-1}\right) \cdot \frac{\sqrt[3]{(n+1)^2} + \sqrt[3]{n+1}\cdot\sqrt[3]{n-1} + \sqrt[3]{(n-1)^2}}{\sqrt[3]{(n+1)^2} + \sqrt[3]{n+1}\cdot\sqrt[3]{n-1} + \sqrt[3]{(n-1)^2}} =$$

$$= \frac{2}{\sqrt[3]{(n+1)^2} + \sqrt[3]{n^2-1} + \sqrt[3]{(n-1)^2}} = y_n$$

$$a_n \geq \sqrt[3]{n-1} - \sqrt[3]{n+1} = \left(\sqrt[3]{n-1} - \sqrt[3]{n+1}\right) \cdot \frac{\sqrt[3]{(n-1)^2} + \sqrt[3]{n-1}\cdot\sqrt[3]{n+1} + \sqrt[3]{(n+1)^2}}{\sqrt[3]{(n-1)^2} + \sqrt[3]{n-1}\cdot\sqrt[3]{n+1} + \sqrt[3]{(n+1)^2}} =$$

$$= \frac{-2}{\sqrt[3]{(n-1)^2} + \sqrt[3]{n^2-1} + \sqrt[3]{(n+1)^2}} = x_n$$

$$\left.\begin{array}{l} 1^\circ \quad x_n \leq a_n \leq y_n, \quad \forall n \\ 2^\circ \quad \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n \end{array}\right\} \Rightarrow \lim_{n\to\infty} a_n = 0$$

h) Izračunati gr. vrijednost

$$\lim_{n\to\infty} \sum_{k=n+1}^{2n} \sqrt{\frac{n+\sqrt{k}}{(-1)^k k^2 - k + 7n^3}}$$

$$a_n = \sum_{k=n+1}^{2n} \sqrt{\frac{n+\sqrt{k}}{7n^3 + (-1)^k \cdot k^2 - k}}$$

$$a_n \leq n\sqrt{\frac{n+\sqrt{2n}}{7n^3 - (2n)^2 - 2n}}$$

$$a_n \geq n\sqrt{\frac{n+\sqrt{n+1}}{7n^3+(2n)^2-(n+1)}}$$

$$x_n = n\sqrt{\frac{n+\sqrt{n+1}}{7n^3+4n^2-n-1}} \quad ; \quad y_n = n\sqrt{\frac{n+\sqrt{2n}}{7n^3-4n^2-2n}}$$

1° $\quad x_n \leq a_n \leq y_n \, , \; \forall n \in \mathbb{N}$

2° $$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\sqrt{\frac{n^3+n^2\sqrt{n+1}}{7n^3+4n^2-n-1}} = \lim_{n\to\infty}\sqrt{\frac{\cancel{n^3}\left(1+\frac{\sqrt{n+1}}{n}\right)}{\cancel{n^3}\left(7+\frac{4}{n}-\frac{1}{n^2}+\frac{1}{n^3}\right)}} = \lim_{n\to\infty}\sqrt{\frac{1+\sqrt{\frac{1}{n}+\frac{1}{n^2}}}{7+\underbrace{\frac{4}{n}}_{0}-\underbrace{\frac{1}{n^2}}_{0}+\underbrace{\frac{1}{n^3}}_{0}}}$$

$$= \frac{1}{\sqrt{7}} = \frac{\sqrt{7}}{7}$$

Iz 1° i 2° slijedi $\lim\limits_{n\to\infty} a_n = \frac{\sqrt{7}}{7}$

$$\lim_{n\to\infty} y_n = \ldots = \frac{\sqrt{7}}{7}$$

21. novembar

Granična vrijednost f-je

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \lim_{x\to 0}\frac{a^x-1}{x} = \ln a \qquad \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1 \qquad \lim_{x\to 0}\frac{\ln(1+x)}{x} = 1 \qquad \lim_{x\to \infty}\left(1+\frac{1}{x}\right)^x = e$$

1. $$\lim_{x\to 3}\frac{x^3-5x^2+6x}{x^2-9} = \lim_{x\to 3}\frac{x(x^2-5x+6)}{x^2-9} = \lim_{x\to 3}\frac{x\cancel{(x-3)}(x-2)}{\cancel{(x-3)}(x+3)} = \frac{3\cdot 1}{6} =$$

2. $$\lim_{x\to +\infty}\frac{2x^4+x^2+2x+3}{3x^4+x^2+x+3} = \lim_{x\to+\infty}\frac{\cancel{x^4}\left(2+\overbrace{\frac{1}{x^2}}^{0}+\overbrace{\frac{2}{x^3}}^{0}+\overbrace{\frac{3}{x^4}}^{0}\right)}{\cancel{x^4}\left(3+\underbrace{\frac{1}{x^2}}_{0}+\underbrace{\frac{1}{x^3}}_{0}+\underbrace{\frac{3}{x^4}}_{0}\right)} = \frac{2}{3}$$

3. $$\lim_{x\to -\infty}\frac{\sqrt{5x^2-7}}{x} = \lim_{x\to -\infty}\frac{\sqrt{x^2\left(5-\frac{7}{x^2}\right)}}{x} = \lim_{x\to -\infty}\frac{|x|\sqrt{5-\frac{7}{x^2}}}{x} = \lim_{x\to -\infty}\frac{-\cancel{x}\sqrt{5-\frac{7}{x^2}}}{\cancel{x}} =$$

$$= -\sqrt{5}$$