Izvodi

$f(x) = x^3$

$f'(x) = 3x^2$

$f'(-1) = 3\cdot(-1)^2 = 3$

$f'(x_0) = \lim_{\Delta x\to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \longrightarrow$ izvod f-je u tački

1. Nađi izvod f-je $f(x) = \sqrt{1+2x}$

Neka je $x_0 \in \mathbb{R}$ proizvoljna tačka

$$f'(x_0) = \lim_{\Delta x\to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\sqrt{1+2(x_0+\Delta x)} - \sqrt{1+2x_0}}{\Delta x}\cdot\frac{\sqrt{1+2(x_0+\Delta x)} + \sqrt{1+2x_0}}{\sqrt{1+2(x_0+\Delta x)} + \sqrt{1+2x_0}}$$

$$= \lim_{\Delta x\to 0} \frac{1+2(x_0+\Delta x) - (1+2x_0)}{\Delta x\left(\sqrt{1+2(x_0+\Delta x)} + \sqrt{1+2x_0}\right)} = \lim_{\Delta x\to 0} \frac{2\Delta x}{\Delta x\left(\sqrt{1+2(x_0+\Delta x)} + \sqrt{1+2x_0}\right)} =$$

$$= \frac{2}{2\sqrt{1+2x_0}} = \frac{1}{\sqrt{1+2x_0}}, \quad 1+2x_0 > 0$$

$$x_0 > -\frac{1}{2}$$

$$\boxed{f'(x) = \frac{1}{\sqrt{1+2x}}, \quad x > -\frac{1}{2}}$$

2. Ispitati diferencijabilnost f-je $f(x) = \frac{1}{2}\sqrt{(x-2)^2}$ u tački $x = 2$.

$f(x) = \frac{1}{2}|x-2|$

$$f'_+(2) = \lim_{\Delta x\to 0+} \frac{f(2+\Delta x) - f(2)}{\Delta x} = \lim_{\Delta x\to 0+} \frac{\frac{1}{2}|2+\Delta x-2| - 0}{\Delta x} = \lim_{\Delta x\to 0+} \frac{\frac{1}{2}\cancel{\Delta x}}{\cancel{\Delta x}} = \frac{1}{2}$$

$$f'_-(2) = \lim_{\Delta x\to 0-} \frac{f(2+\Delta x) - f(2)}{\Delta x} = \lim_{\Delta x\to 0-} \frac{-\frac{1}{2}\cancel{\Delta x}}{\cancel{\Delta x}} = -\frac{1}{2}$$

$f'_+(2) \neq f'_-(2) \Rightarrow$ f-ja f nije diferencijabilna u tački $x = 2$

NEPREKIDNOST $\not\Rightarrow$ DIFERENCIJABILNOST

DIFERENCIJABILNOST $\Rightarrow$ NEPREKIDNOST

3. Odrediti parametre a i b tako da f-ja f bude diferencijabilna na R

$$f(x) = \begin{cases} x^2, & x \geq 1 \\ ax+b, & x < 1 \end{cases}$$

Na intervalima $(-\infty, 1)$ i $(1, +\infty)$ f-ja se poklapa sa polinomima x^2 i $ax+b$ pa je diferencijabilna. Da bi f-ja f bila diferencijabilna u $x=1$ potrebno je, ali ne i dovoljno da u toj tački bude neprekidna, tj. da važi

$$\lim_{x\to 1+} f(x) = \lim_{x \to 1-} f(x) = f(1) \qquad (1)$$

Da bi f-ja bila diferencijabilna u $x=1$ potrebno je i dovoljno da važi:

$$f'_+(1) = f'_-(1) \qquad (2)$$

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} x^2 = 1$$

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} (ax+b) = a+b$$

$$f(1) = 1^2 = 1$$

iz (1) $\Rightarrow$ $a+b=1$

$$f'_+(1) = \lim_{\Delta x\to 0+} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0+} \frac{(1+\Delta x)^2 - 1}{\Delta x} = \lim_{\Delta x\to 0+} \frac{\cancel{\Delta x}(2+\Delta x)}{\cancel{\Delta x}} = 2$$

$$f'_-(1) = \lim_{\Delta x\to 0-} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\Delta x\to 0-} \frac{a(1+\Delta x)+b-1}{\Delta x} = \lim_{\Delta x\to 0-} \frac{a+a\Delta x+b-1}{\Delta x} =$$

$$= \lim_{\Delta x \to 0-} \frac{a\cancel{\Delta x}}{\cancel{\Delta x}} = a$$

(2) $\Rightarrow$ $a = 2$

$a+b=1$

$a=2 \Rightarrow b=-1$

4. Odrediti parametre a i b tako da f-ja f bude dif. u tački $x=0$

$$f(x) = \begin{cases} x + a^2, & x < 0 \\ a \cdot \cos x + b \cdot \sin x, & x \geq 0 \end{cases}$$

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0-} f(x) = f(0) \qquad (1)$$

$$f'_+(0) = f'_-(0) \qquad (2)$$

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} (a\cos x + b\sin x) = a$$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} (x + a^2) = a^2$$

$$f(0) = a \cdot \cos 0 + b \cdot \sin 0 = a$$

$$(1) \Rightarrow \quad a = a^2$$

$$a(1-a) = 0 \iff a = 0 \lor a = 1$$

$$f'_+(0) = \lim_{\Delta x\to 0+} \frac{\overset{f(\Delta x)}{f(0+\Delta x)} - f(0)}{\Delta x} = \lim_{\Delta x\to 0+} \frac{a\cos\Delta x + b\sin\Delta x - a}{\Delta x} =$$

$$= \lim_{\Delta x\to 0+} \left(-a\,\frac{1-\cos\Delta x}{\Delta x} + b \cdot \frac{\sin\Delta x}{\Delta x}\right) =$$

$$= \lim_{\Delta x\to 0+} \left(-a \cdot \underbrace{\cancel{\frac{1-\cos\Delta x}{\Delta x^2}}}_{\frac{1}{2}} \cdot \Delta x + b \cdot \underbrace{\cancel{\frac{\sin\Delta x}{\Delta x}}}_{1}\right) = b$$

$$f'_-(0) = \lim_{\Delta x\to 0-} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x\to 0-} \frac{\Delta x + a^2 - a}{\Delta x} = \lim_{\Delta x\to 0-} \frac{\Delta x + a - a}{\Delta x} = 1$$

$$(2) \Rightarrow \quad b = 1$$

Izvod složene funkcije

$f = f(u) \quad , \quad u = g(x)$

$f'_x = f'_u \cdot u'_x$

$f(x) = x^3$

$f'(x) = 3x^2$

$f(x) = (4x^5 - 2x)^3$

$f'(x) = 3(4x^5 - 2x)^2 \cdot (4x^5 - 2x)'$

$f'(x) = 3(4x^5 - 2x)^2 (20x^4 - 2)$

1. a) $y = \sqrt[3]{1 + x^4} = (1 + x^4)^{\frac{1}{3}}$

$y' = \frac{1}{3}(1 + x^4)^{-\frac{2}{3}} (1 + x^4)'$

$y' = \frac{1}{3}(1 + x^4)^{-\frac{2}{3}} \cdot 4x^3$

$y' = \frac{4}{3} \frac{x^3}{\sqrt[3]{(1 + x^4)^2}}$

12. decembar

b) $y = \cos(2x^3 + 5x)$

$y' = -\sin(2x^3 + 5x) \cdot (6x^2 + 5)$

c) $y = \sin^3 \ln x^5$

$y' = 3\sin^2 \ln x^5 \cdot \cos \ln x^5 \cdot \frac{1}{\cancel{x^5}} \cdot 5\cancel{x^4}$

$y' =$

d) $y = tg(2 \cdot e^{3x})$

$y' = \frac{1}{\cos^2(2e^{3x})} \cdot 2e^{3x} \cdot 3$

e) $y = 3^{2tg^2 x}$

$y = 3^{2tg^2 x} \ln 3 \cdot 4 tg x \cdot \frac{1}{\cos^2 x}$

f) $y = e^{x\ln(x^2+2)}$

$$y' = e^{x\ln(x^2+2)}\left(\ln(x^2+2) + x\cdot\frac{1}{x^2+2}\cdot 2x\right)$$

g) $y = \operatorname{arctg} e^{2x}$

$$y' = \frac{1}{1+e^{4x}}\cdot e^{2x}\cdot 2$$

h) $y = \ln\sqrt{\frac{e^{2x}}{e^{2x}-1}}$

$$y' = \frac{1}{\sqrt{\frac{e^{2x}}{e^{2x}-1}}}\cdot\frac{1}{2\sqrt{\frac{e^{2x}}{e^{2x}-1}}}\cdot\frac{2e^{2x}(e^{2x}-1) - e^{2x}\cdot 2e^{2x}}{(e^{2x}-1)^2} = \frac{1}{\cancel{2}}\cdot\frac{\cancel{e^{2x}-1}}{\cancel{e^{2x}}}\cdot$$

$$\frac{-\cancel{2}\cancel{e^{2x}}}{(e^{2x}-1)^{\cancel{2}}} = -\frac{1}{e^{2x}-1} = \frac{1}{1-e^{2x}}$$

i) $y = \sin^5\left(\ln\sqrt[3]{a^{5x}}\right)$

$$y' = 5\sin^4\left(\ln\sqrt[3]{a^{5x}}\right)\cdot\cos\left(\ln\sqrt[3]{a^{5x}}\right)\cdot\frac{1}{\sqrt[3]{a^{5x}}}\cdot\frac{1}{3}\left(a^{5x}\right)^{-\frac{2}{3}}\cdot a^{5x}\ln a\cdot 5$$

j) $y = x^{a^a} + a^{x^a} + a^{a^x}$, $a > 0$

$$y' = a^a x^{a^a-1} + a^{x^a}\ln a\cdot a\cdot x^{a-1} + a^{a^x}\ln a\cdot a^x\cdot\ln a$$

2. Nađi izvod implicitno zadatih f-ja

a) $xy - y^3 = 1$, Nađi i $y'(0)$

$$xy - y^3 = 1 \quad /'$$

$$1\cdot y + xy' - 3y^2\cdot y' = 0$$

$$y + (x - 3y^2)\cdot y' = 0$$

$$y' = \frac{-y}{x-3y^2}$$

$$y'(x) = -\frac{y_0}{x_0 - 3y_0^2}, \quad y = f(x_0)$$

$x_0 = 0$

$x_0\cdot y_0 - y_0^3 = 1$

$0\cdot y_0 - y_0^3 = 1$

$y_0^3 = -1 \Rightarrow y_0 = -1$

$$y'(0) = \frac{-(-1)}{0-3(-1)}$$

$$y'(0) = -\frac{1}{3}$$

b) $xy - \sqrt{xy^2+6} = 0$, i odrediti $y'(x_0)$ za $x_0 = 3$ i $y_0 > 0$

$$y + xy' - \frac{1}{2\sqrt{xy^2+6}} \cdot (y^2 + x\,2yy') = 0$$

$$y' = \frac{y\left(y - 2\sqrt{xy^2+6}\right)}{2x\left(\sqrt{xy^2+6} - y\right)}$$

$$y'(3) = \frac{1\cdot\left(1 - 2\sqrt{3\cdot 1^2+6}\right)}{2\cdot 3\left(\sqrt{3\cdot 1^2+6} - 1\right)} = \frac{-5}{12}$$

$$x_0 y_0 - \sqrt{x_0 y_0^2 + 6} = 0$$

$$3y_0 = \sqrt{3y_0^2+6} \quad /^2$$

$$9y_0^2 = 3y_0^2 + 6$$

$$y_0^2 = 1 \quad \Rightarrow \quad y_0 = \pm 1 \qquad y_0 = 1 > 0$$

3. F-ja y definisana sa $\operatorname{arctg}\left(\frac{y}{x}\right) = \frac{1}{2}\ln(x^2+y^2)$ zadovoljava j-nu

$y'\cdot(x-y) = x+y$. Dokazati

R $\quad \operatorname{arctg}\left(\frac{y}{x}\right) = \frac{1}{2}\ln(x^2+y^2) \quad /'$

$$\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{y}{x}\right)' = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot(2x+2yy')$$

$$\frac{x^2}{x^2+y^2}\cdot\left(\frac{y}{x}\right)' = \frac{x+yy'}{x^2+y^2}$$

$$\cancel{x^2}\cdot\frac{y'x - y}{\cancel{x^2}} = x + yy'$$

$$xy' - y = x + yy'$$

$$y'(x-y) = x+y \qquad \text{q.e.d}$$

4. Naći prvi izvod sl. f-ja

a) $y = x^x$

$\ln y = \ln x^x$

$\ln y = x \ln x \quad /'$

$\frac{1}{y} \cdot y' = \ln x + x \cdot \frac{1}{x}$

$\frac{1}{y} y' = \ln x + 1$

$y' = y(\ln x + 1)$

$y' = x^x (\ln x + 1)$

b) $y = x^{\ln x}$

$\ln y = \ln x^{\ln x}$

$\ln y = \ln x \cdot \ln x = (\ln x)^2 \quad /'$

$\frac{1}{y} \cdot y' = 2 \ln x \cdot \frac{1}{x}$

$y' = y \cdot \frac{2}{x} \cdot \ln x$

$y' = x^{\ln x} \cdot \frac{2}{x} \cdot \ln x$

c) $y = (\sin x)^{1+\cos^2 x}$

$\ln y = (1+\cos^2 x) \ln(\sin x)$

$\frac{1}{y} \cdot y' = 2\cos x(-\sin x) \cdot \ln(\sin x) + (1+\cos^2 x) \cdot \frac{1}{\sin x} \cdot \cos x$

$y' = (\sin x)^{1+\cos^2 x} \left(-2 \sin x \cos x \cdot \ln(\sin x) + (1+\cos^2 x) \operatorname{ctg} x \right)$

d) $x^y = y^x$

$y \ln x = x \ln y \quad /'$

$y' \ln x + y \cdot \frac{1}{x} = \ln y + x \cdot \frac{1}{y} \cdot y'$

$\left(\ln x - \frac{x}{y} \right) \cdot y' = \ln y - \frac{y}{x}$

$y' = \dfrac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}$

5. Naći prvi izvod f-je zadate parametarskim j-nama $\begin{cases} x = 2t+3t^2 \\ y = t^2+2t^3 \end{cases}$, pokazati da f-ja zadovoljava j-nu $2y'^3 + y'^2 - y = 0$

R: $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ $\quad y'_x = \dfrac{g'_t}{f'_t}$ $\qquad y'_x = \dfrac{y'_t}{x'_t} = \dfrac{2t+6t^2}{2+6t} = \dfrac{t\cancel{(2+6t)}}{\cancel{2+6t}} = t$

Za tačku $M(5,3)$ odgovarajuće t je $t=1$ $\qquad y'(5) = 1$

$$2y'^3 + y'^2 - y = 0$$
$$2t^3 + t^2 - (t^2+2t^3) = 0$$
$$0 = 0$$

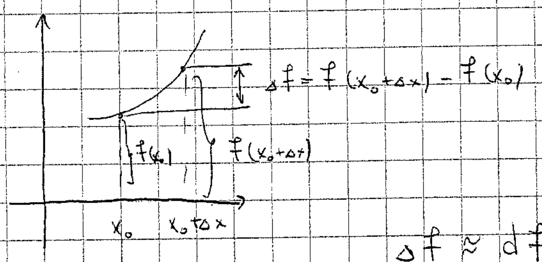
6. Koristeći izvod inverzne f-je naći izvod f-je $f(x) = \arcsin x$

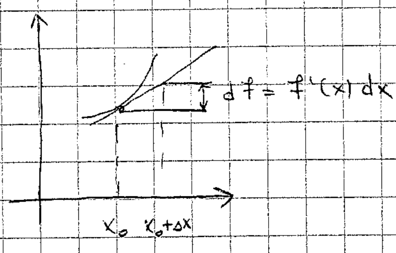
$y = f(x) = \arcsin x$ $\qquad f'(x) = \dfrac{1}{(f^{-1}(y))'} = \dfrac{1}{(\sin y)'} = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1-\sin^2 y}} =$

$y=f(x) \Rightarrow$ $f^{-1}(y) = x$

$f^{-1}(y) = \sin y$ $\qquad = \dfrac{1}{\sqrt{1-x^2}}$

Diferencijal i njegova primjena





$$\Delta f \approx df$$

1. Primjenom diferencijala približno izračunati $(5,123)^2$

R: Neka je $f(x) = x^2$. Tražimo $f(5,123)$

$x_0 = 5$ $\qquad$ Tražimo $f(x_0 + \Delta x)$

$\Delta x = 0,123$

$\Delta f \approx df$

$$f(x_0+\Delta x) - f(x_0) \approx f'(x)\,dx$$

$$f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\,dx$$

$$f(5{,}123) \approx f(5) + f'(5)\cdot 0{,}123$$

$$(5{,}123)^2 \approx 25 + 10\cdot 0{,}123$$

$$(5{,}123)^2 \approx 26{,}23$$

$f(5) = 5^2 = 25$

$f'(x) = 2x$

$f'(5) = 2\cdot 5 = 10$

2. Primjenom diferencijala približno izračunati $\sqrt[3]{2{,}97}$

R: Neka je $f(x) = \sqrt[3]{x}$. Tražimo $f(2{,}97)$

$x_0 = 3$

$\Delta x = -0{,}03$

Tražimo $f(x_0+\Delta x)$

$f(3) = \sqrt[3]{3}$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

$f'(3) = \frac{1}{3\sqrt[3]{9}}$

$$f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\,dx$$

$$f(2{,}97) \approx f(3) + f'(3)\cdot(-0{,}03)$$

$$\sqrt[3]{2{,}97} \approx \sqrt[3]{3} + \frac{1}{3\sqrt[3]{9}}\cdot(-0{,}03)$$

3. Primjenom diferencijala približno izračunati $\sqrt{\frac{2{,}037^2-1}{2{,}037^2+1}}$

R: Neka je $f(x) = \sqrt{\frac{x^2-1}{x^2+1}}$. Tražimo $f(2{,}037)$

Neka je $x_0 = 2$ i $\Delta x = 0{,}037$. Tražimo $f(x_0+\Delta x)$

$$f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\,dx$$

$$f(2{,}037) \approx f(2) + f'(2)\cdot 0{,}037$$

$$\sqrt{\frac{2{,}037^2-1}{2{,}037^2+1}} \approx \sqrt{\frac{3}{5}} + \frac{4}{5}\sqrt{\frac{5}{3}}\cdot 0{,}037$$

$f(2) = \sqrt{\frac{4-1}{4+1}} = \sqrt{\frac{3}{5}}$

$f'(x) = \frac{1}{2\sqrt{\frac{x^2-1}{x^2+1}}}\cdot\frac{2x(x^2+1)-(x^2-1)\cdot 2x}{(x^2+1)^2}$

$= \frac{1}{2}\sqrt{\frac{x^2+1}{x^2-1}}\cdot\frac{\cancel{4}^{\,2}x}{(x^2+1)^2}$

$f'(2) = 2\sqrt{\frac{5}{3}}\cdot\frac{2}{25}$

4. Naći dy za:

a) $y = 2^x - 3^{-x} + \sqrt{x}$

$dy = f'(x)\,dx$

$dy = \left(2^x \ln 2 - 3^{-x} \ln 3 \,(-1) + \frac{1}{2\sqrt{x}}\right) dx$

b) $y = \ln\left(\cos \frac{1}{x}\right) - \frac{1}{x}$

$dy = \left(\frac{1}{\cos \frac{1}{x}}\left(-\sin \frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) - \left(-\frac{1}{x^2}\right)\right) dx$

$dy = \frac{1}{x^2}\left(\operatorname{tg} \frac{1}{x} + 1\right) dx$

Izvodi i diferencijali

1. Naći y'', y''' i y^{IV} ako je $y = x \ln x$

R// $y' = \ln x + x \cdot \frac{1}{x} = \ln x + 1$

$y'' = (y')' = \frac{1}{x}$

$y''' = (y'')' = -\frac{1}{x^2}$

$y^{IV} = (y''')' = \frac{2}{x^3}$

2. Naći n-ti izvod f-ja

a) $y = x^n$, b) $y = \frac{1}{x+1}$

a) $y' = n x^{n-1}$

$y'' = n(n-1)\, x^{n-2}$

$y''' = n(n-1)(n-2)\, x^{n-3}$

$y^{(k)} = n(n-1) \cdots (n-(k-1)) \cdot x^{n-k}$, $k \leq n$

$y^{(n)} = n(n-1) \cdots 1 \cdot x^0 = n!$

$y^{(n+1)} = 0$ $y^{(k)} = 0$ za $k > n$

b) $y = (x+1)^{-1}$

$y' = -\frac{1}{(x+1)^2}$

$y'' = \frac{2}{(x+1)^3}$ $y''' = -\frac{6}{(x+1)^4}$

$y^{(n)} = (-1)^n \cdot \frac{n!}{(x+1)^{n+1}}$

3. Naći $\dfrac{d^2y}{dx^2}$ za sl. f-je:

a) $\operatorname{arctg} y = y - x$

$$d^2y = y'' dx^2 \qquad y'' = \frac{d^2y}{dx^2}$$

$$\operatorname{arctg} y = y - x \quad /'$$

$$\frac{1}{1+y^2} \cdot y' = y' - 1$$

$$\left(1 - \frac{1}{1+y^2}\right) y' = 1$$

$$\frac{y^2}{1+y^2}\, y' = 1$$

$$y' = \frac{1+y^2}{y^2}$$

$$y' = 1 + \frac{1}{y^2}$$

$$y'' = (y')' = \left(1 + \frac{1}{y^2}\right)'_x = -2y^{-3}\, y'$$

$$y'' = -\frac{2}{y^3}\, y'$$

$$y'' = -\frac{2}{y^3} \cdot \left(\frac{1+y^2}{y^2}\right)$$

$$y'' = -2 \cdot \frac{1+y^2}{y^5}$$

b) $\begin{cases} x = 2t - t^2 \\ y = 3t - t^3 \end{cases}$

$$y'_x = \frac{y'_t}{x'_t} = \frac{3 - 3t^2}{2 - 2t} = \frac{3(1-t)(1+t)}{2(1-t)} = \frac{3}{2}(1+t)$$

$$y''_x = \left(\frac{3}{2}(1+t)\right)'_x = \left(\frac{3}{2}(1+t)\right)'_t t'_x = \frac{3}{2} \cdot \frac{1}{x'_t} = \frac{3}{2} \cdot \frac{1}{2-2t} = \frac{3}{4} \cdot \frac{1}{1-t}$$

4. Neka je $x = x(y)$ inverzna f-ja f-je $y = y(x)$. Dokazati da važi

$$\frac{d^3y}{dx^3} \cdot \left(\frac{dx}{dy}\right)^2 + \frac{d^3x}{dy^3} \cdot \left(\frac{dy}{dx}\right)^2 + 3 \cdot \frac{d^2y}{dx^2} \cdot \frac{d^2x}{dy^2} = 0$$

$$\frac{dy}{dx} = y' \qquad \frac{dx}{dy} = x'_y = \frac{1}{y'_x}$$

$$(y')^{-1} \qquad (-1)(y')^{-2}\, y'$$

$$\left(\frac{1}{y'_x}\right)'_x \cdot x'_y = -\frac{1}{y'^2} \cdot y'' \cdot \frac{1}{y'} = -\frac{y''}{y'^3}$$

$$\frac{d^2y}{dx^2} = y'' \qquad \frac{d^2x}{dy^2} = x''_y = (x'_y)'_y = \left(\frac{1}{y'_x}\right)'_y = \cancel{(-1) \cdot y'^{-2} \cdot y''} = \cancel{\frac{-y''}{y'^2}}$$

$$\frac{d^3y}{dx^3} = y''' \qquad \frac{d^3x}{dy^3} = (x''_y)'_y = \left(\frac{-y''}{y'^3}\right)'_y = \cancel{-\frac{y''' \cdot y'^2 - y'' \cdot 2y' \cdot y''}{y'^4}} =$$

$$= \cancel{-\frac{y' \cdot y''' - 2y''^2}{y'^3}}$$

$$\left(\frac{-y''}{y'^3}\right)'_x \cdot x'_y = -\frac{y''' \cdot y'^3 - y'' \cdot 3y'^2 \cdot y''}{y'^6} \cdot \frac{1}{y'}$$

$$-\frac{y' \cdot y''' - 3y''^2}{y'^5}$$

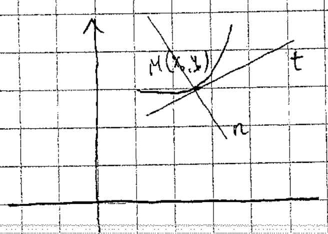
$$y''' \cdot \frac{1}{y'^2} + \left(\frac{2y''^2}{y'^3} - \frac{y'''}{y'^2}\right) \cdot y'^2 + 3 \cdot y'' \cdot \left(-\frac{y''}{y'^2}\right) = 0$$

$$\frac{y'''}{y'^2} + \left(-\frac{y' y''' - 3y''^2}{y'^5}\right) \cdot y'^2 + 3 \cdot y'' \cdot \left(-\frac{y''}{y'^3}\right) = 0$$

$$\frac{y'''}{y'^2} - \frac{y'''}{y'^2} + 3\frac{y''^2}{y'^3} - 3\frac{y''^2}{y'^3} = 0$$

$$0 = 0$$

Tangente i normale



$k_t = f'(x_0)$

$M(x_0, y_0)$, $y_0 = f(x_0)$

$t: \; y - y_0 = f'(x_0)(x - x_0)$

$n: \; y - y_0 = -\frac{1}{f'(x_0)}(x - x_0) \quad \Rightarrow \quad k_n = -\frac{1}{f'(x_0)}$

$k_t \cdot k_n = -1$

1. Napisati j-ne tangente i normale na grafik f-je $f(x) = \sqrt{x}$ u tački za koju je $x_0 = 4$

$y_0 = f(x_0) = f(4) = \sqrt{4} = 2$

$M(4, 2)$

$f'(x) = \frac{1}{2\sqrt{x}}$

$t: \; y - y_0 = f'(x_0)(x - x_0)$

$y - 2 = f'(4)(x - 4)$

$y - 2 = \frac{1}{4}(x - 4)$

$y = \frac{x}{4} + 1$

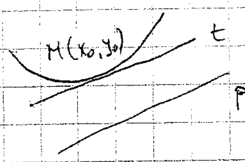
$n: \; y - 2 = -\frac{1}{f'(4)}(x - 4)$

$y - 2 = -\frac{1}{\frac{1}{4}}(x - 4)$

$y - 2 = -4x + 16$

$y = -4x + 18$

2. Na krivoj $y = x^2 + 3x - 4$ postaviti tangentu koja je paralelna pravoj $y = 3x - 3$



$k_t = f'(x_0)$ $\qquad$ $p \parallel t \Rightarrow f'(x_0) = 3$

$k_p = 3$

$f'(x) = 2x + 3$

$2x_0 + 3 = 3$

$x_0 = 0 \Rightarrow y_0 = x_0^2 + 3x_0 - 4$

$y_0 = -4$

$t: y - (-4) = 3(x - 0)$

$M(0, -4)$

$t: y = 3x - 4$

3. Napisati j-ne normala grafika f-je $y = \dfrac{2}{x^2+1}$ koje prolaze kroz koordinatni početak.

Neka je normala postavljena u tački $M(x_0, y_0)$

$y_0 = f(x_0)$

$f'(x) = \dfrac{-2}{(x^2+1)^2} \cdot 2x = \dfrac{-4x}{(x^2+1)^2}$

$y_0 = \dfrac{2}{x_0^2+1}$

$n: y - y_0 = \dfrac{-1}{f'(x_0)}(x - x_0)$ $\qquad$ $f'(x_0) = \dfrac{-4x_0}{(x_0^2+1)^2}$

$y - \dfrac{2}{x_0^2+1} = \dfrac{(x_0^2+1)^2}{4x_0}(x - x_0)$

$O(0,0) \in n \Longrightarrow \dfrac{-2}{x_0^2+1} = \dfrac{(x_0^2+1)^2}{4x_0}(-x_0)$

$\dfrac{2}{x_0^2+1} = \dfrac{(x_0^2+1)^2}{4}$

$(x_0^2+1)^3 = 8$

$x_0^2 + 1 = 2$

$x_0^2 = 1$

$x_0 = \pm 1$

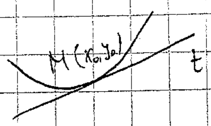
$x_0 = 1$ $\qquad$ $x_0 = -1$

$y_0 = \dfrac{2}{1+1} = 1$ $\qquad$ $M_1(1, 1)$ $\qquad$ $y_0 = \dfrac{2}{1+1} = 1$ $\qquad$ $M_2(-1$

$n_1: y-1 = 1\cdot(x-1)$ $\qquad$ $n_2: y-1 = -1(x-(-1))$

$y = x$ $\qquad M_1$ $\qquad$ $y = -x$ $\qquad M_2$

4. Odrediti parametar k tako da prava $y = kx+1$ bude tangenta krive $y^2 = 4x$ i naći tačku dodira.



Ako je prava tangenta onda je $k = f'(x_0)$

$$y^2 = 4x \quad /'_x$$

$$2yy' = 4$$

$$y' = \frac{2}{y}$$

$y_0^2 = 4x_0$

$y'(x_0) = \dfrac{2}{y_0}$

$M(x_0, y_0)$

$t: y = \dfrac{2}{y_0}x + n$ $\qquad$ $t: y - y_0 = \dfrac{2}{y_0}(x - x_0)$

$y = \dfrac{2}{y_0}x + y_0 - \dfrac{2}{y_0}x_0$

$M \in t \Rightarrow y_0 = \dfrac{2}{y_0}x_0 + 1$

$\cancel{y_0 = \dfrac{2}{y_0}\cdot\dfrac{y_0^2}{4}}$

$\cancel{y_0 = \dfrac{2}{y_0}\cdot\dfrac{y_0^2}{4} + y_0 - \dfrac{2}{y_0}\cdot\dfrac{y_0^2}{4}}$

$\cancel{y_0 = y_0}$

$\cancel{y_0 =}$

$$y_0 = \frac{\cancel{2}}{\cancel{y_0}}\cdot\frac{y_0^{\cancel{2}}}{\cancel{4}_2} + 1$$

$$\frac{y_0}{2} = 1$$

$$y_0 = 2 \Rightarrow x_0 = 1$$

$$k = \frac{2}{y_0} = \frac{2}{2} = 1$$

19. decembar

1. Napisati j-nu tangente i normale krive $\begin{cases} x = \dfrac{2t}{t+2} \\ y = \dfrac{t}{t-1} \end{cases}$ u tački $M(1,1)$

$t: y-1 = y'(1)(x-1)$

$n: y-1 = -\dfrac{1}{y'(1)}(x-1)$

$y'_x = \dfrac{y'_t}{x'_t}$

$$y'_t = \frac{t-1-t}{(t-1)^2} = \frac{-1}{(t-1)^2}$$

$$x'_t = \frac{2(t+2)-2t}{(t+2)^2} = \frac{4}{(t+2)^2}$$

$$y'_x = \frac{\frac{-1}{(t-1)^2}}{\frac{4}{(t+2)^2}} = -\frac{(t+2)^2}{4(t-1)^2}$$

Nađimo t za tačku M

$$x=1 \Rightarrow \frac{2t}{t+2} = 1 \qquad y'(1) = -\frac{16}{4} = -4$$

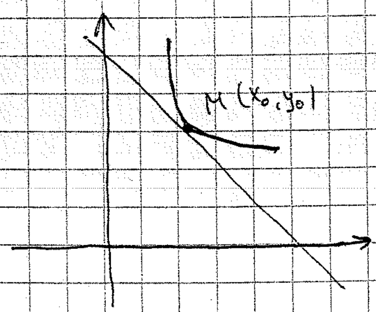
$$2t = t+2$$

$$t = 2$$

$$t: \; y-1 = -4(x-1) \qquad n: \; y-1 = -\frac{1}{-4}(x-1)$$

$$y = -4x+5 \qquad y = \frac{1}{4}x + \frac{3}{4}$$

2. Data je kriva $y=(\sqrt{a}-\sqrt{x})^2$. Tangenta te krive siječe koordinatne ose u tačkama A i B. Dokazati da je $OA + OB = const.$



Neka je tangenta postavljena u proizvoljnoj tački $M(x_0, y_0)$ date krive

$$t: \; y - y_0 = f'(x_0)(x-x_0)$$

$$y = (\sqrt{a}-\sqrt{x})^2$$

1) $$y' = 2(\sqrt{a}-\sqrt{x})\left(-\frac{1}{2\sqrt{x}}\right)$$

$$y' = -\frac{\sqrt{a}-\sqrt{x}}{\sqrt{x}}$$

$$y'(x_0) = -\frac{\sqrt{a}-\sqrt{x_0}}{\sqrt{x_0}}$$

$$M(x_0, y_0) \in k \Rightarrow y_0 = (\sqrt{a}-\sqrt{x_0})^2$$

$$y - (\sqrt{a}-\sqrt{x_0})^2 = -\frac{\sqrt{a}-\sqrt{x_0}}{\sqrt{x_0}}(x-x_0)$$

Presjek tangente i Ox ose $(y=0)$

$$-(\sqrt{a}-\sqrt{x_0})^2 = -\frac{\sqrt{a}-\sqrt{x_0}}{\sqrt{x_0}}(x-x_0)$$

$$\sqrt{a}-\sqrt{x_0} = \frac{x-x_0}{\sqrt{x_0}}$$

$$\sqrt{a}\cdot\sqrt{x_0} - \cancel{x_0} = x - \cancel{x_0}$$

$$x = \sqrt{a}\cdot\sqrt{x_0} \qquad A(\sqrt{a}\cdot\sqrt{x_0}, 0)$$

Presjek sa Oy osom $(x=0)$

$$y-(\sqrt{a}-\sqrt{x_0})^2=(\sqrt{a}-\sqrt{x_0})\sqrt{x_0}$$

$$y=a-2\sqrt{a}\sqrt{x_0}+x_0+\sqrt{a}\sqrt{x_0}-x_0$$

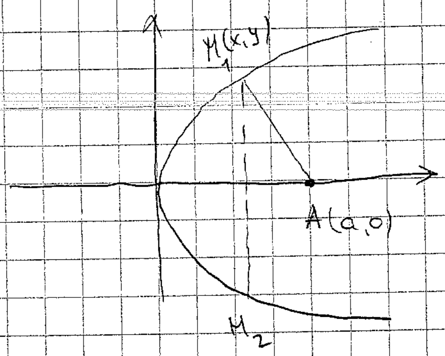
$$y=a-\sqrt{a}\sqrt{x_0}$$

$$B\left(0,\ a-\sqrt{a}\sqrt{x_0}\right)$$

$$OA+OB=a$$

Ekstremne vrijednosti f-je

1. Na paraboli $y^2=2px$ naći tačku najbližu tački $A(a,0)$



Udaljenost između M i A

$$d(M,A)=\sqrt{(x-a)^2+y^2}$$

$$y^2=2px$$

$$f(x)=d(M,A)=\sqrt{(x-a)^2+2px}$$

Tražimo minimum f-je f

$$f'(x)=\frac{1}{2\sqrt{(x-a)^2+2px}}\cdot\left(2(x-a)+2p\right)$$

$$f'(x)=\frac{x-a+p}{\sqrt{(x-a)^2+2px}}$$

$$f'(x)=0 \iff x-a+p=0$$
$$\iff x=a-p$$

$$f'(x)>0 \iff x-a+p>0$$
$$\iff x>a-p$$

$$f'(x)<0 \iff x-a+p<0$$
$$\iff x<a-p$$

$$f_{min}=f(a-p)=$$
$$=\sqrt{p^2+2p(a-p)}=$$
$$=\sqrt{2ap-p^2}$$

f' $+$

u $x=a-p$ f-ja postiže minimum

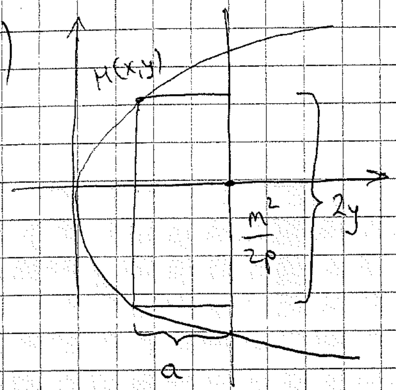
$$y^2 = 2px$$

$$y^2 = 2p(a-p) \qquad M_1\left(a-p, \sqrt{2p(a-p)}\right), \quad M_2\left(a-p, -\sqrt{2p(a-p)}\right)$$

$$y = \pm\sqrt{2p(a-p)}$$

2. Naći $\square$ maksimalne površine ograničen parabolom $y^2 = 2px$ i pravom $x = \frac{m^2}{2p}$ ako jedna stranica pravougaonika leži na zadatoj pravoj

1)



$$a = \frac{m^2}{2p} - x$$

$$b = 2y$$

2) $P = a \cdot b = \left(\frac{m^2}{2p} - x\right) 2y$

$$y^2 = 2px \Rightarrow x = \frac{y^2}{2p}$$

$$P(y) = \left(\frac{m^2}{2p} - \frac{y^2}{2p}\right) 2y$$

$$P(y) = \frac{1}{p}\left(m^2 y - y^3\right)$$

3) Tražimo max f-je $P(y)$

$$P'(y) = \frac{1}{p}\left(m^2 - 3y^2\right)$$

$$P'(y) = 0 \Leftrightarrow m^2 - 3y^2 = 0$$

$$\Leftrightarrow y^2 = \frac{m^2}{3}$$

$$\Leftrightarrow y = \pm\frac{m}{\sqrt{3}}$$

$y = \frac{m}{\sqrt{3}}$ (prvi kvadrant)

$$y = \frac{m\sqrt{3}}{3}$$

$$\frac{m^2}{3} = 2px \Rightarrow x = \frac{m^2}{6p}$$

$$M\left(\frac{m^2}{6p}, \frac{m\sqrt{3}}{3}\right)$$

4) $P''(y) = \frac{1}{p}(-6y)$

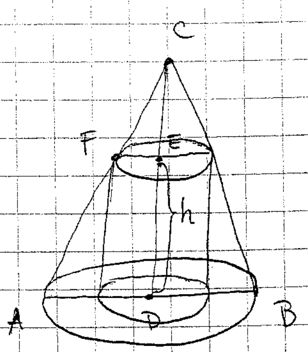
$$P''\left(\frac{m\sqrt{3}}{3}\right) = \frac{1}{p}\left(-6\,\frac{m\sqrt{3}}{3}\right) < 0$$

$\Rightarrow$ f-ja P u tački $y = \frac{m\sqrt{3}}{3}$ postiže maksimum. Taj maksimum je

$$P_{max} = P\left(\frac{m\sqrt{3}}{3}\right) = \frac{1}{p}\left(m^2 \cdot \frac{m\sqrt{3}}{3} - \frac{m^3 \cdot \cancel{3}\sqrt{3}}{\cancel{27}\,9}\right) = \frac{1}{p}\, m^3\, \frac{2\sqrt{3}}{9}$$

(kupa)

3. U dati pravi konus poluprečnika R i visine H upisati cilindar maksimalne zapremine.



$CD = H$

$AD = R$

$\triangle ADC \sim \triangle FEC \quad \left(\measuredangle ADC = \measuredangle FEC = \frac{\pi}{2} \wedge \measuredangle ACD = \measuredangle FCE \right)$

$AD : CD = FE : CE$

$R : H = r : (H-h)$

$$r = \frac{R(H-h)}{H}$$

V - zapremina cilindra

$V = B \cdot h = r^2 \pi h$

$$V_{(h)} = \frac{R^2 (H-h)^2}{H^2} \pi \cdot h$$

$$V(h) = \frac{R^2 \pi}{H^2} \left(H^2 h - 2Hh^2 + h^3 \right)$$

$$V'(h) = \frac{R^2 \pi}{H^2} \left(H^2 - 4Hh + 3h^2 \right)$$

$$V'(h) = 0 \iff 3h^2 - 4Hh + H^2 = 0$$

$$h_{1/2} = \frac{4H \pm \sqrt{16H^2 - 12H^2}}{6} = \frac{4H \pm 2H}{6}$$

$$\cancel{h_1 = H} \qquad h_2 = \frac{H}{3}$$

$$r = \frac{R \frac{2}{3} \cancel{H}}{\cancel{H}} \Rightarrow r = \frac{2}{3} R$$

$$V''(h) = \frac{R^2 \pi}{H^2} \left(-4H + 6h \right)$$

$$V''\left(\frac{H}{3}\right) = \frac{R^2 \pi}{H^2} \left(-4H + \overset{2}{\cancel{6}} \cdot \frac{H}{\cancel{3}} \right) = -\frac{2R^2 \pi}{H} < 0 \Rightarrow$$

$\Rightarrow$ f-ja V u tački $h = \frac{H}{3}$ postiže maksimum.

$$V_{max} = V\left(\frac{H}{3}\right) = \frac{R^2 \frac{4}{9} \cancel{H^2}}{\cancel{H^2}} \cdot \pi \cdot \frac{H}{3} = \frac{4}{27} R^2 \pi H \qquad \frac{4}{9} \left(\frac{1}{3} R^2 \pi H \right)$$

1. U zatvorenoj figuri koju obrazuju krive $y = 2 - x^2$ i $y = |x|$ upisati pravougaonik maksimalne površine tako da mu 2 tjemena leže na datoj paraboli.

$y = 2 - x^2$

$2 - x^2 = 0$

$x^2 = 2$

$x = \pm\sqrt{2}$

$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$\begin{cases} y = 2 - x^2 \\ y = x, \ x \geq 0 \end{cases}$

$2 - x^2 = x$

$x^2 + x - 2 = 0, \quad x \geq 0$

$x_{1,2} = \frac{-1 \pm 3}{2}$

$x_1 = 1 \ \wedge \ \cancel{x_2 = -2}$

$\Downarrow$

$y_1 = 1$

$M_1(1, 1)$

$\begin{cases} y = 2 - x^2 \\ y = -x, \ x < 0 \end{cases}$

$2 - x^2 = -x$

$x^2 - x - 2 = 0, \quad x < 0$

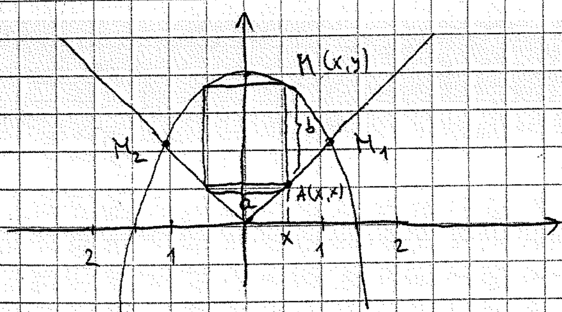
$x_{1,2} = \frac{1 \pm 3}{2}$

$\cancel{x_1 = 2} \ \wedge \ x_2 = -1$

$\Downarrow$

$y_2 = 1$

$M_2(-1, 1)$



$a = 2x$

$b = y - x$

$M(x, y) \in$ paraboli $\Rightarrow y = 2 - x^2$

$b = 2 - x^2 - x$

$P = ab = 2x(2 - x^2 - x) = -2x^3 - 2x^2 + 4x$

$P' = -6x^2 - 4x + 4 = 2(-3x^2 - 2x + 2)$

$P'(x) = 0 \iff -3x^2 - 2x + 2 = 0$

$x_{1,2} = \frac{1 \pm \sqrt{7}}{-3}$

$\cancel{x_1 = \frac{1 + \sqrt{7}}{-3}} \qquad x_2 = \frac{1 - \sqrt{7}}{-3} > 0$

$x = \frac{\sqrt{7} - 1}{3}$

$P''(x) = 2(-6x - 2)$

$$P''(x) = 2(-6x-2)$$

$P''\left(\frac{\sqrt{7}-1}{3}\right) = -4\left(3\,\frac{\sqrt{7}-1}{3}+1\right) < 0 \implies$ funkcija $P(x)$ postiže max u

tački $x = \frac{\sqrt{7}-1}{3}$. Taj maksimum je

$$P_{max} = P\left(\frac{\sqrt{7}-1}{3}\right) = -2\left(\left(\frac{\sqrt{7}-1}{3}\right)^3 + \left(\frac{\sqrt{7}-1}{3}\right)^2 - 2\,\frac{\sqrt{7}-1}{3}\right) = \ldots$$

$$y = 2 - x^2 = 2 - \left(\frac{\sqrt{7}-1}{3}\right)^2 = \frac{18-7+2\sqrt{7}-1}{9} = \frac{10+2\sqrt{7}}{9}$$

$$M\left(\frac{\sqrt{7}-1}{3}, \frac{10+2\sqrt{7}}{9}\right)$$