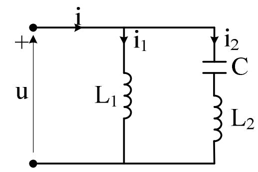
ANALIZA KOLA U USTALJENOM PSEUDOPERIODIČNOM I SLOŽENOPERIODIČNOM REŽIMU

1. Dato je reaktivno kolo prikazano na slici:



napon je složenoperiodičan

$e = \sqrt{2}U^{(1)}\cos\omega t + \sqrt{2}U^{(3)}\cos 3\omega t$

a) Odrediti učestanost napona kola da ukupna struja bude prostoperiodična učestanosti osnovnog harmonika napona.

b) Odrediti trenutnu vrijednost struje u kalemu induktivnosti L_1, klir faktor ove struje i napona U.

c) Odrediti aktivnu, reaktivnu i prividnu snagu i snagu izobličenja u grani sa kondenzatorom.

Rješenje

a) ako se prvo odredi ulazna impedansa

$$\underline{Z}_{ul}^{(k)} = \frac{jk\omega L_1\left(jk\omega L_2 + \dfrac{1}{jk\omega C}\right)}{jk\omega L_1 + jk\omega L_2 + \dfrac{1}{jk\omega C}} = \frac{jk\omega L_1\left(1 - k^2\omega^2 L_2 C\right)}{1 - k^2\omega^2 C\left(L_1 + L_2\right)}$$

poznato je da važi

$$\underline{I}^{(k)} = \frac{\underline{U}^{(k)}}{\underline{Z}_{ul}^{(k)}} \quad \rightarrow \quad \underline{I}^{(1)} = \frac{\underline{U}^{(1)}}{\underline{Z}_{ul}^{(1)}} \qquad \underline{I}^{(3)} = \frac{\underline{U}^{(3)}}{\underline{Z}_{ul}^{(3)}}$$

Da bi ulazna struja bila prostoperiodična sa učestanošću osnovnog harmonika napona potrebno je da je struja trećeg harmonika nula, tj. odgovarajuća ulazna impedansa mora biti beskonačno velika.

$$\underline{Z}_{ul}^{(3)} = \infty \ \rightarrow \quad \frac{j3\omega L_1\left(1 - 9\omega^2 L_2 C\right)}{1 - 9\omega^2 C\left(L_1 + L_2\right)} \rightarrow \infty$$

$$1 - 9\omega^2 C\left(L_1 + L_2\right) = 0 \rightarrow \omega = \frac{1}{3\sqrt{C\left(L_1 + L_2\right)}}$$

b) $$\underline{I}_1^{(k)} = \frac{\underline{U}^{(k)}}{jk\omega L_1} \rightarrow \underline{I}_1^{(1)} = \frac{\underline{U}^{(1)}}{j\omega L_1} = \frac{U^{(1)}}{\omega L_1}\underline{/-90^\circ} \qquad \underline{I}_1^{(3)} = \frac{\underline{U}^{(3)}}{j3\omega L_1} = \frac{U^{(3)}}{3\omega L_1}\underline{/-90^\circ}$$

sada je vremenski oblik struje

$$i_1(t) = i_1^{(1)}(t) + i_1^{(3)}(t) = \sqrt{2}\frac{U^{(1)}}{\omega L_1}\cos\left(\omega t - \frac{\pi}{2}\right) + \sqrt{2}\frac{U^{(3)}}{3\omega L_1}\cos\left(3\omega t - \frac{\pi}{2}\right)$$

klir faktor je prema definiciji

$$k = \frac{\sqrt{\left(I^{(2)}\right)^2 + \left(I^{(3)}\right)^2 + \left(I^{(4)}\right)^2 + \ldots + \left(I^{(k)}\right)^2}}{I} = \frac{\sqrt{\left(I^{(2)}\right)^2 + \left(I^{(3)}\right)^2 + \left(I^{(4)}\right)^2 + \ldots + \left(I^{(k)}\right)^2}}{\sqrt{\left(I^{(0)}\right)^2 + \left(I^{(1)}\right)^2 + \left(I^{(2)}\right)^2 + \ldots + \left(I^{(k)}\right)^2}}$$

klir faktor struje i_1

$$k_{i_1} = \frac{I_1^{(3)}}{\sqrt{\left(I_1^{(1)}\right)^2 + \left(I_1^{(3)}\right)^2}} = \frac{\dfrac{U^{(3)}}{3\omega L_1}}{\sqrt{\left(\dfrac{U^{(1)}}{\omega L_1}\right)^2 + \left(\dfrac{U^{(3)}}{3\omega L_1}\right)^2}} = \frac{U^{(3)}}{\sqrt{9\left(U^{(1)}\right)^2 + \left(U^{(3)}\right)^2}}$$

klir faktor napona u

$$k_u = \frac{U^{(3)}}{\sqrt{\left(U^{(1)}\right)^2 + \left(U^{(3)}\right)^2}}$$

c) U grani sa kondenzatorom aktivna snaga je jednaka nuli jer nema aktivnih potrošača (otpornosti). Kako bi se odredila snaga, potrebno je prvo odrediti impedansu posmatrane grane

$$\underline{Z}_2^{(k)} = jk\omega L_2 + \frac{1}{jk\omega C} = \frac{jk^2\omega^2 L_2 C - j}{k\omega C} = -j\frac{1 - k^2\omega^2 L_2 C}{k\omega C}$$

ako se zamjeni ranije dobijena vrijednost ω

$$\underline{Z}_2^{(1)} = -j\frac{1 - \frac{1}{9C(L_1+L_2)}L_2 C}{\frac{1}{3\sqrt{C(L_1+L_2)}}C} = -j\frac{\frac{9(L_1+L_2)-L_2}{9C(L_1+L_2)}}{\frac{1}{3\sqrt{C(L_1+L_2)}}} = -j\frac{9L_1+8L_2}{3\sqrt{C(L_1+L_2)}}$$

$$\underline{Z}_2^{(3)} = -j\frac{1 - 9\frac{1}{9C(L_1+L_2)}L_2 C}{3\frac{1}{3\sqrt{C(L_1+L_2)}}C} = -j\frac{\frac{9C(L_1+L_2)-9L_2C}{9C(L_1+L_2)}}{3\frac{1}{3\sqrt{C(L_1+L_2)}}C} = -j\frac{3L_1}{\sqrt{C(L_1+L_2)}}$$

reaktivna snaga posmatrane grane kola je

$$\underline{Q}_2^{(1)} = -\frac{\left(U^{(1)}\right)^2}{Z_2^{(1)}} \qquad \underline{Q}_2^{(3)} = -\frac{\left(U^{(3)}\right)^2}{Z_2^{(3)}}$$

ukupna reaktivna snaga je

$$Q_2 = Q_2^{(1)} + Q_2^{(3)} = -\frac{3\sqrt{C(L_1+L_2)}}{9L_1+8L_2}\left[U^{(1)}\right]^2 - \frac{\sqrt{C(L_1+L_2)}}{L_1}\left[U^{(3)}\right]^2$$

Prividna snaga je prema definiciji

$$S_2 = UI_2 = \sqrt{\left(U^{(1)}\right)^2 + \left(U^{(3)}\right)^2}\sqrt{\left(\frac{U^{(1)}}{Z_2^{(1)}}\right)^2 + \left(\frac{U^{(3)}}{Z_2^{(3)}}\right)^2}$$

Snaga izobličenja je

$$D_2 = \sqrt{S_2^2 - Q_2^2} = \sqrt{\left(\left(U^{(1)}\right)^2 + \left(U^{(3)}\right)^2\right)\left(\left(\frac{U^{(1)}}{Z_2^{(1)}}\right)^2 + \left(\frac{U^{(3)}}{Z_2^{(3)}}\right)^2\right) - \left(\frac{\left(U^{(1)}\right)^2}{Z_2^{(1)}} + \frac{\left(U^{(3)}\right)^2}{Z_2^{(3)}}\right)^2} =$$

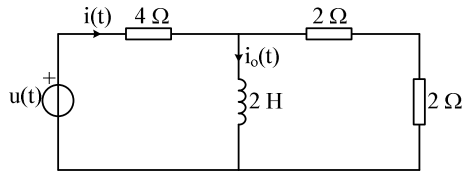
$$= \sqrt{\frac{\left(U^{(1)}\right)^4}{\left(Z_2^{(1)}\right)^2} + \frac{\left(U^{(1)}\right)^2\left(U^{(3)}\right)^2}{\left(Z_2^{(3)}\right)^2} + \frac{\left(U^{(3)}\right)^2\left(U^{(1)}\right)^2}{\left(Z_2^{(1)}\right)^2} + \frac{\left(U^{(3)}\right)^4}{\left(Z_2^{(3)}\right)^2} - \frac{\left(U^{(1)}\right)^4}{\left(Z_2^{(1)}\right)^2} - 2\frac{\left(U^{(1)}\right)^2\left(U^{(3)}\right)^2}{Z_2^{(1)}Z_2^{(3)}} - \frac{\left(U^{(3)}\right)^4}{\left(Z_2^{(3)}\right)^2}} =$$

$$= \sqrt{\frac{\left(U^{(1)}\right)^2\left(U^{(3)}\right)^2}{\left(Z_2^{(3)}\right)^2} + \frac{\left(U^{(3)}\right)^2\left(U^{(1)}\right)^2}{\left(Z_2^{(1)}\right)^2} - 2\frac{\left(U^{(1)}\right)^2\left(U^{(3)}\right)^2}{Z_2^{(1)}Z_2^{(3)}}} = \sqrt{\left(\frac{U^{(1)}U^{(3)}}{Z_2^{(3)}} - \frac{U^{(3)}U^{(1)}}{Z_2^{(1)}}\right)^2} = \left|\frac{U^{(1)}U^{(3)}}{Z_2^{(3)}} - \frac{U^{(3)}U^{(1)}}{Z_2^{(1)}}\right| =$$

$$= U^{(1)}U^{(3)}\left(\frac{\sqrt{C(L_1+L_2)}}{3L_1} - \frac{3\sqrt{C(L_1+L_2)}}{9L_1+8L_2}\right)$$

2. Naći izraz za struju $i_0(t)$, ako je napon izvora dat u obliku Furijeovog reda

$$u(t) = 1 + \sum_{n=1}^{\infty}\frac{2(-1)^n}{1+n^2}(\cos nt - n\sin nt).$$



Rješenje

Izraz za ulazni napon može se transformisati jer važi

$$k_1\cos x + k_2 \sin x = \sqrt{k_1^{\,2}+k_2^{\,2}}\cos\left(x + arctg\,\frac{-k_2}{k_1}\right)$$

pa je napon

$$u(t) = 1 + \sum_{n=1}^{\infty}\frac{2(-1)^n}{\sqrt{1+n^2}}\cos(nt + arctg\,n)$$

ili ako se razvije red

$$u(t) = 1 - 1.414\cos(t+45°) + 0.8944\cos(2t+63.45°) - 0.6345\cos(3t+71.56°) + ...$$

Struja i_o se može naći ako se prvo odredi struja i, pa onda struja i_o na osnovu strujnog razdjelnika. Ulazna impedansa n-tog harmonika je

$$\underline{Z}^{(n)} = 4 + \frac{j\omega_n 2\cdot 4}{4 + j\omega_n 2} = \frac{8 + j\omega_n 8}{2 + j\omega_n}$$

struja je

$$\underline{I}^{(n)} = \frac{\underline{U}^{(n)}}{\underline{Z}^{(n)}} = \frac{2 + j\omega_n}{8 + j\omega_n 8}\underline{U}^{(n)} \rightarrow \underline{I}_o^{(n)} = \frac{4}{4 + j\omega_n 2}\underline{I}^{(n)} = \frac{4}{4 + j\omega_n 2}\frac{2 + j\omega_n}{8 + j\omega_n 8}\underline{U}^{(n)} = \frac{1}{4 + j\omega_n 4}\underline{U}^{(n)}$$

Kako se vidi iz definicije reda, važi

$$\omega_n = n \rightarrow \underline{I}_o^{(n)} = \frac{\underline{U}^{(n)}}{4 + jn4} = \frac{\underline{U}^{(n)}}{4\sqrt{1+n^2}\,\underline{/\,arctg\,n}}$$

$$n = 0 \rightarrow \underline{I}_o = \frac{1}{4}$$

Za n-ti harmonik

$$\underline{U}^{(n)} = \frac{2(-1)^n}{\sqrt{1+n^2}}\underline{/\,arctg\,n} \rightarrow \underline{I}_o^{(n)} = \frac{\frac{2(-1)^n}{\sqrt{1+n^2}}\underline{/\,arctg\,n}}{4\sqrt{1+n^2}\,\underline{/\,arctg\,n}} = \frac{2(-1)^n}{4(1+n^2)} = \frac{(-1)^n}{2(1+n^2)}\underline{/0°}$$

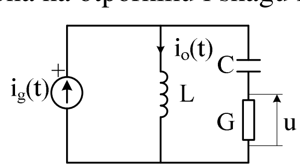
sada je vremenski oblik struje

$$i_o(t) = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n}{2(1+n^2)}\cos nt \text{ [A]}$$

3. Za kolo prema slici poznato je: ω, G, $i_g(t) = I^{(1)}\cos\omega t + I^{(3)}\cos 3\omega t$

a) Odrediti parametre L i C tako da osnovni harmonik napona otpornika ne zavisi od G i da amplituda trećeg harmonika struje u otporniku iznosi 90% od amplitude trećeg harmonika struje izvora.

b) Izračunati trenutnu vrijednost napona na otporniku i snagu izobličenja kondenzatora.



Rješenje

Napon na otporniku je, koristeći strujni razdjelnik

$$\underline{U}^{(k)}=\frac{jk\omega L}{jk\omega L+\frac{1}{G}+\frac{1}{jk\omega C}}\underline{I}_g^{(k)}\cdot\frac{1}{G}=\frac{jk\omega L}{jk\omega GL+1+\frac{G}{jk\omega C}}\underline{I}_g^{(k)}=\frac{jk\omega L}{1+jG\left(k\omega L-\frac{1}{k\omega C}\right)}\underline{I}_g^{(k)}$$

Prvi uslov zadatka je da osnovni harmonik napona na otporniku ne zavisi od G, a to je ispunjeno samo u slučaju kada važi

$$\omega L-\frac{1}{\omega C}=0 \qquad (*)$$

Drugi uslov je da je amplituda trećeg harmonika struje u otporniku iznosi 90% trećeg harmonika struje izvora.

$$\underline{I}^{(k)}=G\underline{U}^{(k)}=G\frac{jk\omega L}{1+jG\left(k\omega L-\frac{1}{k\omega C}\right)}\underline{I}_g^{(k)}$$

pa se može pisati

$$0.9I_g^{(3)}=\frac{3\omega GL}{\sqrt{1+G^2\left(3\omega L-\frac{1}{3\omega C}\right)^2}}I_g^{(3)} \qquad (**)$$

Riješavajući sistem koji čine relacije (*) i (**) dolazi se do

$$C=\frac{2G}{\omega} \qquad L=\frac{1}{2G\omega}$$

b) Trenutna vrijednost napona

$$\underline{U}^{(k)}=\frac{j\frac{k}{2G}}{1+j\frac{1}{2}\left(k-\frac{1}{k}\right)}\underline{I}_g^{(k)}\rightarrow\underline{U}^{(1)}=j\frac{1}{2G}\underline{I}_g^{(1)}\rightarrow u^{(1)}(t)=\frac{1}{2G}I^{(1)}\cos\left(\omega t-\frac{\pi}{2}\right)$$

$$\underline{U}^{(3)}=j\frac{3}{2G}\frac{1}{1+j\frac{4}{3}}\underline{I}_g^{(3)}=\frac{3}{2G}\frac{3}{4-j3}\underline{I}_g^{(3)}=0.9\frac{\underline{I}_g^{(3)}}{G}e^{jarctg\frac{3}{4}}\rightarrow u^{(3)}(t)=0.9\frac{I^{(3)}}{G}\cos\left(3\omega t+arctg\frac{3}{4}\right)$$

$$u(t)=u^{(1)}(t)+u^{(3)}(t)$$

Snaga izobličenja je

$$D_C=\sqrt{S^2-P^2-Q^2}=\frac{1}{2}\left|U_C^{(1)}I_C^{(3)}-U_C^{(3)}I_C^{(1)}\right|=\frac{1}{2}\left|\frac{1}{\omega C}I_C^{(1)}I_C^{(3)}-\frac{1}{3\omega C}I_C^{(3)}I_C^{(1)}\right|=\frac{1}{3\omega C}I_C^{(1)}I_C^{(3)}$$

zna se da važi

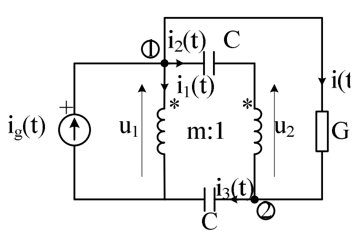
$$I_C^{(1)}=I_G^{(1)}=GU^{(1)}=\frac{I_g^{(1)}}{2}$$

$$I_C^{(3)}=I_G^{(3)}=GU^{(3)}=0.9I_g^{(3)}$$

pa je

$$D_C=\frac{1}{3\cdot 26}\frac{I_g^{(1)}}{2}0.9I_g^{(3)}=\frac{3I_g^{(1)}I_g^{(3)}}{406}$$

4. Ako je struja strujnog generatora u kolu prema slici $i_g(t) = I_o+I_1\cos\omega t$ odrediti trenutnu vrijednost ustaljenog odziva u otporniku ako je m=1/2.



Rješenje

Cilj je odrediti struju i(t). Sa slike važi:

$$\frac{\underline{U}_1}{\underline{U}_2}=m \qquad \frac{\underline{I}_1}{\underline{I}_2}=-\frac{1}{m}$$

a kako je m poznato onda

$$\underline{U}_2=2\underline{U}_1 \qquad \underline{I}_1=-2\underline{I}_2$$

Iz KZS za čvor 1:

$$\underline{I}_1+\underline{I}_2+\underline{I}=\underline{I}_g \rightarrow -2\underline{I}_2+\underline{I}_2+\underline{I}=\underline{I}_g \rightarrow \underline{I}_2=\underline{I}-\underline{I}_g$$

Iz KZS za čvor 2:

$$\underline{I}_3-\underline{I}_2-\underline{I}=0 \rightarrow \underline{I}_3=2\underline{I}-\underline{I}_g$$

Naponi $\underline{U}_1$ i $\underline{U}_2$ su

$$\underline{U}_1=\frac{\underline{I}}{G}+\frac{1}{j\omega C}\underline{I}_3=\frac{\underline{I}}{G}+\frac{1}{j\omega C}\left(2\underline{I}-\underline{I}_g\right)$$

$$\underline{U}_2=\frac{\underline{I}}{G}-\frac{1}{j\omega C}\underline{I}_2=\frac{\underline{I}}{G}-\frac{1}{j\omega C}\left(\underline{I}-\underline{I}_g\right)$$

Vrativši se na početnu relaciju za napone idealnog transformatora

$$\underline{U}_2=2\underline{U}_1 \rightarrow \frac{\underline{I}}{G}-\frac{1}{j\omega C}\left(\underline{I}-\underline{I}_g\right)=2\left[\frac{\underline{I}}{G}+\frac{1}{j\omega C}\left(2\underline{I}-\underline{I}_g\right)\right]$$

$$\frac{\underline{I}}{G}+5\frac{1}{j\omega C}\underline{I}-\frac{3}{j\omega C}\underline{I}_g=0 \rightarrow \underline{I}=\frac{\frac{3}{j\omega C}}{\frac{1}{G}+\frac{5}{j\omega C}}\underline{I}_g=\frac{3G}{5G+j\omega C}\underline{I}_g$$

Struja k-tog harmonika u ustaljenom režimu je:

$$\underline{I}^{(k)}=\frac{3G}{5G+jk\omega C}\underline{I}_g^{(k)} \rightarrow k=0 \qquad \underline{I}^{(o)}=\frac{3}{5}\underline{I}_g^{(o)}=\frac{3}{5}I_o$$

$$k=1 \qquad \underline{I}^{(1)}=\frac{3G}{5G+j\omega C}\underline{I}_g^{(1)}=\frac{3I_1}{\sqrt{25+\frac{1}{G^2}C^2\omega^2}}e^{-jarctg\frac{\omega C}{5G}}$$

sada je tražena struja u vremenskom domenu

$$i\left(t\right)=\frac{3}{5}I_o+\frac{3I_1}{\sqrt{25+\frac{1}{G^2}C^2\omega^2}}\cos\left(\omega t-arctg\frac{\omega C}{5G}\right)$$

TRIGONOMETRIJSKI FURIJEOV RED

Prema definiciji funkcija se razlaže u Furijeov red prema izrazu sa koeficijentima:

$$f\left(t\right)=\frac{a_o}{2}+\sum_{n=1}^{\infty}\left(a_n\cos n\omega_o t+b_n\sin n\omega_o t\right)$$

$$a_o=\frac{2}{T}\int_0^T f\left(t\right)dt$$

$$a_n=\frac{2}{T}\int_0^T f\left(t\right)\cos n\omega_o tdt$$

$$b_n=\frac{2}{T}\int_0^T f\left(t\right)\sin n\omega_o tdt$$

Alternativna forma je amplitudno fazna forma:

$$f(t)=\frac{a_o}{2}+\sum_{n=1}^{\infty}A_n\cos(n\omega_o t+\phi_n)$$

$$A_n=\sqrt{a_n^{\,2}+b_n^{\,2}} \qquad \phi_n=-arctg\frac{b_n}{a_n} \qquad \underline{A_n}=a_n+jb_n$$

$$^{*}\ a\cos\alpha+b\sin\alpha=\sqrt{a^2+b^2}\cos\left(\alpha-\arctan\frac{b}{a}\right)$$

Grafik A_n po $n\omega_o$ naziva se amplitudski spektar, a grafik ϕ_n po $n\omega_o$ naziva se fazni spektar. Oba grafika čine frekvencijski spektar. Neki korisni integrali:

$$\int\cos at dt=\frac{1}{a}\sin at \qquad \int\sin at dt=-\frac{1}{a}\cos at$$

$$\int t\cos at dt=\frac{1}{a^2}\cos at+\frac{1}{a}t\sin at \qquad \int t\sin at dt=\frac{1}{a^2}\sin at-\frac{1}{a}t\cos at$$

Određivanje faznog stava kompleksnog broja

$$z=x+jy \qquad \theta=arctg\frac{y}{x}$$

$$z=-x+jy \qquad \theta=180°-arctg\frac{y}{x}$$

$$z=-x-jy \qquad \theta=180°+arctg\frac{y}{x}$$

$$z=x-jy \qquad \theta=360°-arctg\frac{y}{x}$$

Neki korisni identiteti:

$$\cos 2n\pi=1 \qquad \cos n\pi=(-1)^n$$

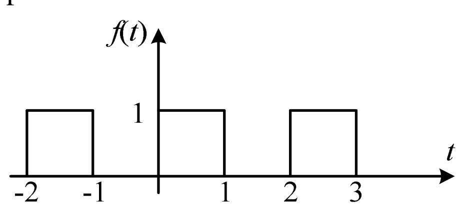
$$\sin 2n\pi=0 \qquad \sin n\pi=0$$

$$\cos\frac{n\pi}{2}=\begin{cases}(-1)^{\frac{n}{2}}, & n\ parno\\ 0, & neparno\end{cases} \qquad \sin\frac{n\pi}{2}=\begin{cases}(-1)^{\frac{(n-1)}{2}}, & n\ parno\\ 0, & neparno\end{cases}$$

$$e^{j2n\pi}=1 \qquad e^{jn\pi}=(-1)^n$$

$$e^{jn\frac{\pi}{2}}=\begin{cases}(-1)^{\frac{n}{2}}, & n\ parno\\ j(-1)^{\frac{(n-1)}{2}}, & neparno\end{cases}$$

1. Razviti u Furijeov red signal prikazan na slici:



Rješenje

Funkcija se može zapisati kao:

$$f(t)=\begin{cases}1, & 0<t<1\\ 0, & 1<t<2\end{cases} \qquad T=2\rightarrow\omega_o=\frac{2\pi}{T}=\pi$$

Sada su koeficijenti Furijeovog reda

$$a_o = \frac{2}{T}\int_0^T f(t)\,dt = \frac{2}{2}\left(\int_0^1 1dt + \int_1^2 0dt\right) = 1$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_o t dt = \frac{2}{2}\left(\int_0^1 1\cdot\cos n\omega_o t dt + \int_1^2 0\cdot\cos n\omega_o t dt\right) = \frac{1}{n\pi}\sin n\pi = 0$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_o t dt = \frac{2}{2}\left(\int_0^1 1\cdot\sin n\omega_o t dt + \int_1^2 0\cdot\sin n\omega_o t dt\right) = -\frac{1}{n\pi}(\cos n\pi - 1) = \frac{1}{n\pi}\left[1-(-1)^n\right]$$

Za b_n može se pisati

$$b_n = \begin{cases} 2/n\pi, & n\, neparno \\ 0, & n\ parno \end{cases}$$

Sada je razvoj funkcije u red

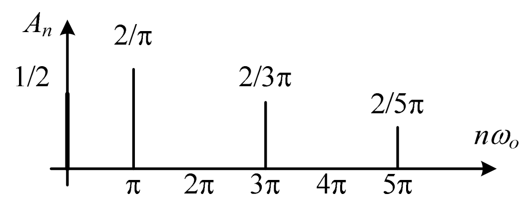
$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sin\pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t + ... = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{1}{n\pi}\sin n\pi t,$$ gdje je n=2k-1

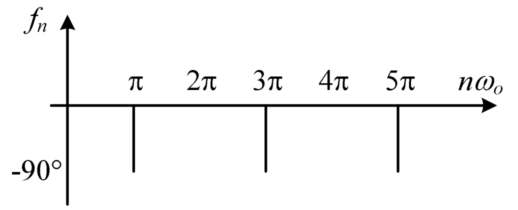
Amplitudski i fazni spektar se dobijaju prema:

$$A_n = \sqrt{{a_n}^2 + {b_n}^2} = |b_n| = \begin{cases} 2/n\pi, & n\ neparno \\ 0, & n\ parno \\ 1/2, & n = 0 \end{cases}$$

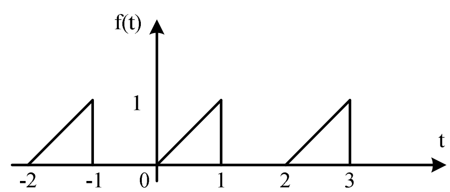
$$\phi_n = -arctg\frac{b_n}{a_n} = \begin{cases} -90°, & n\ neparno \\ 0, & n\ parno \end{cases}$$

a grafička predstava





2. Razviti u Furijeov red periodičnu funkciju prikazanu na slici:



<u>Rješenje</u>

Periodična funkcija može se napisati kao:

$$f(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases} \qquad T = 2 \rightarrow \omega_o = \frac{2\pi}{T} = \pi$$

Sada su koeficijenti Furijeovog reda

$$a_o = \frac{2}{T}\int_0^T f(t)\,dt = \frac{2}{2}\left(\int_0^1 tdt + \int_1^2 0dt\right) = \frac{1}{2}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_o t dt = \frac{2}{2}\left(\int_0^1 t\cdot\cos n\pi t dt + \int_1^2 0\cdot\cos n\pi t dt\right) = \frac{1}{n^2\pi^2}(\cos n\pi - 1) = \frac{(-1)^n - 1}{n^2\pi^2}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_o t dt = \frac{2}{2}\left(\int_0^1 t\cdot\sin n\pi t dt + \int_1^2 0\cdot\sin n\pi t dt\right) = -\frac{\cos n\pi}{n\pi} = -\frac{(-1)^n}{n\pi} = \frac{(-1)^{n+1}}{n\pi}$$

sada je razvoj funkcije u red

$$f(t)=\frac{1}{4}+\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2\pi^2}\cos n\pi t+\frac{(-1)^{n+1}}{n\pi}\sin n\pi t\right]$$

Može se primjetiti da važi, za parne harmonike

$$a_n=0 \qquad b_n=-\frac{1}{n\pi}$$

$$A_n\underline{/\phi_n}=a_n-jb_n=0+j\frac{1}{n\pi}$$

$$A_n=\sqrt{{a_n}^2+{b_n}^2}=|b_n|=\frac{1}{n\pi},\quad n=2,4,...$$

$$\phi_n=90°,\quad n=2,4,...$$

za neparne harmonike

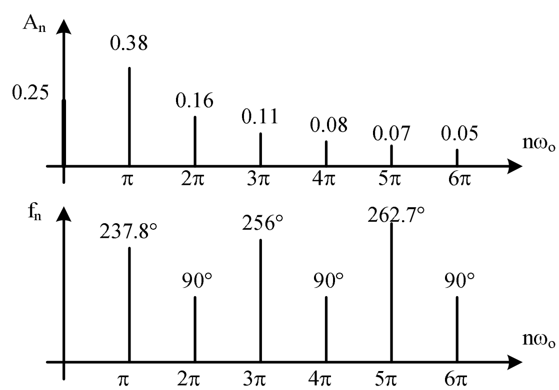
$$a_n=-\frac{2}{n^2\pi^2} \qquad b_n=\frac{1}{n\pi}$$

$$A_n\underline{/\phi_n}=a_n-jb_n=-\frac{2}{n^2\pi^2}-j\frac{1}{n\pi}$$

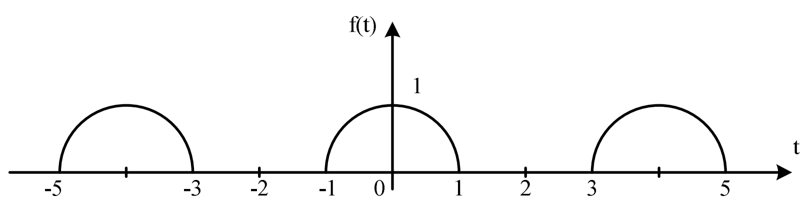
$$A_n=\sqrt{{a_n}^2+{b_n}^2}=\frac{1}{n^2\pi^2}\sqrt{4+n^2\pi^2},\quad n=1,3,...$$

$$\phi_n=180°+arctg\frac{n\pi}{2},\quad n=1,3,...$$

Grafički prikaz je



3. Odrediti Furijeov red funkcije prikazane na slici



Rješenje

Ovo je parna funkcija (simetrična je u odnosu na ordinatu), pa je $b_n=0$. Sa slike se vidi da je T=4, pa slijedi

$$\omega=\frac{2\pi}{T}=\frac{\pi}{2}$$

Funkcija se može zapisati u obliku

$$f(t)=\begin{cases}0, & -2<t<-1\\ \cos\frac{\pi}{2}t, & -1<t<1\\ 0, & 1<t<2\end{cases}$$

Sada su koeficijenti reda:

$$a_o = 2\frac{2}{T}\int_0^{T/2} f(t)\,dt = \frac{4}{4}\left(\int_0^1 \cos\frac{\pi}{2}t\,dt + \int_1^2 0\,dt\right) = \frac{2}{\pi}$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_o t\,dt = \frac{4}{4}\left(\int_0^1 \cos\frac{\pi}{2}t\cos n\frac{\pi}{2}t\,dt + \int_1^2 0\cdot\cos n\pi t\,dt\right) =$$

$$=^{\left[\cos a\cos b = \frac{1}{2}\left[\cos(a+b)+\cos(a-b)\right]\right]} = \frac{1}{2}\left(\int_0^1\left[\cos\frac{\pi}{2}(n+1)t + \cos\frac{\pi}{2}(n-1)t\right]dt\right)$$

$$n = 1 \to a_1 = \frac{1}{2}\left(\int_0^1\left[\cos\pi t + 1\right]dt\right) = \frac{1}{2}$$

$$n > 1 \to a_n = \frac{1}{\pi(n+1)}\sin\frac{\pi}{2}(n+1) + \frac{1}{\pi(n-1)}\sin\frac{\pi}{2}(n-1)$$

$$n - neparno \to (\text{n+1})\ \text{i}\ (\text{n-1})\ \text{su parni} \to \sin\frac{\pi}{2}(n+1) = \sin\frac{\pi}{2}(n-1) = 0 \to a_n = 0$$

$$n - parno \to \sin\frac{\pi}{2}(n+1) = -\sin\frac{\pi}{2}(n-1) = (-1)^{\frac{n}{2}} \to a_n = \frac{(-1)^{\frac{n}{2}}}{\pi(n+1)} + \frac{-(-1)^{\frac{n}{2}}}{\pi(n-1)} = \frac{-2(-1)^{\frac{n}{2}}}{\pi(n^2-1)}$$

sada je funkcija

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\cos\frac{\pi}{2}t - \frac{2}{\pi}\sum_{\substack{n=2\\ n\ parno}}^{\infty}\frac{(-1)^{\frac{n}{2}}}{(n^2-1)}\cos\frac{n\pi}{2}t \quad \text{gdje je } n=2,4,6,\ldots$$

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\cos\frac{\pi}{2}t - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(4k^2-1)}\cos k\pi t \qquad \text{gdje je } n=2k$$

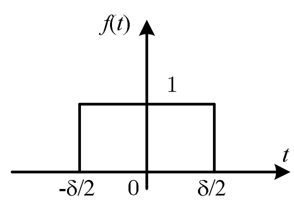
FURIJEOVA TRANSFORMACIJA

1. Naći Furijeovu transformaciju funkcije $f(t)=e^{-at}u(t)$, $a>0$ gdje je $u(t)=h(t)$ Hevisajdova funkcija.

$$F(j\omega) = \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_0^{\infty} e^{-at}e^{-j\omega t}dt = \int_0^{\infty} e^{-(a+j\omega)t}dt = \frac{1}{-(a+j\omega)}e^{-(a+j\omega)t}\Big/_0^{\infty} = \frac{1}{a+j\omega}$$

2. Naći Furijeovu transformaciju pravougaonog impulsa definisanog kao:

$$f(t) = \begin{cases} A, & \frac{-\delta}{2} < t < \frac{\delta}{2} \\ 0, & |t| > \frac{\delta}{2} \end{cases}$$



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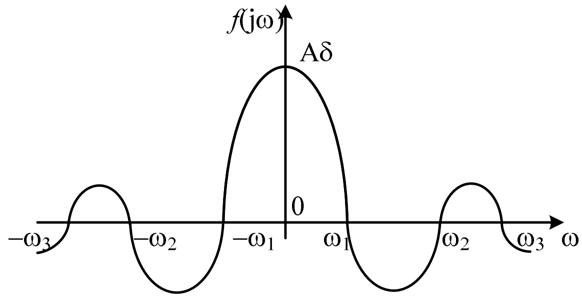
$$F(j\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt = \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} Ae^{-j\omega t}dt = \frac{A}{-j\omega}e^{-j\omega t}\Big/_{-\delta/2}^{\delta/2} = \frac{2A}{j\omega}\frac{e^{j\omega\frac{\delta}{2}} - e^{-j\omega\frac{\delta}{2}}}{2} = \frac{2A}{\omega}\sin\omega\frac{\delta}{2}$$

jer se zna da važi

$$\frac{e^{j\omega t} - e^{-j\omega t}}{2} = \sin\omega t \qquad \frac{e^{j\omega t} + e^{-j\omega t}}{2} = \cos\omega t$$

Kao što se vidi, Furijeova transformacija je čisto realna, pa je moguće nacrtati njen grafik (spektar signala).

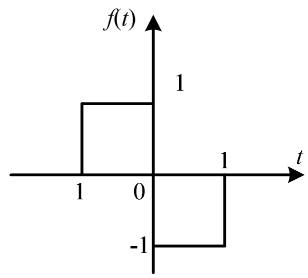


$$F(j\omega)=\frac{2A}{\omega}\sin\omega\frac{\delta}{2}=2A\frac{\delta}{2}\frac{\sin\frac{\omega\delta}{2}}{\frac{\omega\delta}{2}}=A\delta\frac{\sin\frac{\omega\delta}{2}}{\frac{\omega\delta}{2}}$$

$$\omega_1=\frac{2\pi}{\delta}\qquad \omega_2=\frac{4\pi}{\delta}\qquad \omega_3=\frac{6\pi}{\delta}$$

3. Naći FT ako je

$$f(t)=\begin{cases}1, & -1<t<0\\ -1, & 0<t<1\\ 0, & ostalo\,t\end{cases}$$



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$$F(j\omega)=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt=\int_{-1}^{0}e^{-j\omega t}dt-\int_{-1}^{0}e^{-j\omega t}dt=\frac{1}{-j\omega}e^{-j\omega t}\,/_{-1}^{0}-\frac{1}{-j\omega}e^{-j\omega t}\,/_{0}^{1}=$$

$$=\frac{1}{-j\omega}\left(1-e^{j\omega}\right)-\frac{1}{-j\omega}\left(e^{-j\omega}-1\right)=2\frac{j}{\omega}\left(2-\left(e^{j\omega}+e^{-j\omega}\right)\right)=\frac{4j}{\omega}(1-\cos\omega)$$

4.Naći Furijeovu transformaciju funkcije $f(t)=e^{-a|t|}$, a>0

R

$$|t|=\begin{cases}t, & t>0\\ -t, & t<0\end{cases}$$

$$F(j\omega)=\int_{-\infty}^{0}e^{at}e^{-j\omega t}dt+\int_{0}^{-\infty}e^{-at}e^{-j\omega t}dt=\frac{2a}{a^2+\omega^2}$$

5. Naći f(t) ako je $F(j\omega)=\begin{cases}1, & -1<\omega<1\\ 0, & |\omega|>1\end{cases}$

R

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(j\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-1}^{1}1\cdot e^{j\omega t}d\omega=\frac{1}{2\pi}\frac{1}{jt}e^{j\omega t}\,/_{-1}^{1}=\frac{1}{2\pi}\frac{1}{jt}\left(e^{jt}-e^{-jt}\right)=\frac{1}{\pi}\frac{\sin t}{t}$$