1. Posmatra se kolo sa akumulisanom energijom. Odrediti napon $u(t)$ u datom kolu ako je:

a) $u_g(t) = \frac{2}{3}e^{-3t}$

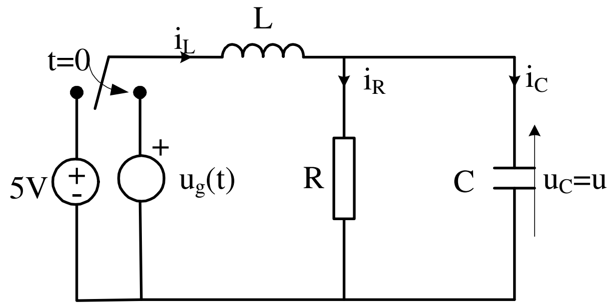
b) $u_g(t) = \frac{2}{3}e^{-2t}$

c) $u_g(t) = 3t^2$

d) $u_g(t) = \frac{5}{3}\cos t$

e) $u_g(t) = \frac{2}{3}e^{t}\sin 2t$

R=1Ω, L=4/3H, C=1/4F.



Rješenje

U svim slučajevima za $t<0$ je $u_g(t)=5V$. U stacionarnom stanju se određuju početni uslovi:

$$i_C(0^-)=0 \Rightarrow i(0^-)=i_L(0^-)=i_L(0^+)=\frac{u_g}{R}=5A$$

$$u_C(0^-)=u_C(0^+)=u_g=5V$$

Za $t\geq 0$ može se odrediti:

$$i_L=i_R+i_C,\quad i_C=C\frac{du}{dt}\quad \text{i}\quad i_R=\frac{u}{R}$$

$$i_L=\frac{u}{R}+C\frac{du}{dt}$$

$$u_g=L\frac{di_L}{dt}+u=LC\frac{d^2u}{dt^2}+\frac{L}{R}\frac{du}{dt}+u$$

$$\Rightarrow\left(D^2+\frac{1}{RC}D+\frac{1}{LC}\right)u(t)=\frac{1}{LC}u_g(t), t\geq 0$$

Odnosno, za zadate vrijednosti R, L i C je :

$$\left(D^2+4D+3\right)u(t)=3u_g(t)$$

Odnosno, rješenje karakteristične jednačine je:

$$s^2+4s+3=0$$

$$s_{1/2}=\frac{-4\pm\sqrt{16-12}}{2}\Rightarrow s_1=-3, s_2=-1$$

Dalje rješavanje zavisi od slučaja do slučaja.

a) $u_g(t)=\frac{2}{3}e^{-3t}, t\geq 0$

Rješenje se traži kao zbir homogenog partikularnog:

$$u(t)=u_h(t)+u_p(t)$$

$$u_h(t)=Ae^{s_1t}+Be^{s_2t}$$

$$u_p(t)=Cte^{-3t}$$

Za određivanje $u_p(t)$ dalje nam je potrebno:

$u_p'(t) = -3Cte^{-3t} + Ce^{-3t}$

$u_p''(t) = 9Cte^{-3t} - 3Ce^{-3t} - 3Ce^{-3t} = 9Cte^{-3t} - 6Ce^{-3t}$

$\left(D^2 + 4D + 3\right)u_p(t) = 3 \cdot \frac{2}{3}e^{-3t}$

$\cancel{9Cte^{-3t}} - 6Ce^{-3t} \cancel{-12Cte^{-3t}} + 4Ce^{-3t} + \cancel{3Cte^{-3t}} = 2e^{-3t}$

$-2Ce^{-3t} = 2e^{-3t} \Rightarrow C = -1$

Odakle je

$u_p(t) = -te^{-3t}$

$u(t) = Ae^{-3t} + Be^{-t} - te^{-3t}$

Na osnovu početnih uslova je:

$u\left(0^+\right) = 5V = A + B$

$i_L\left(0^+\right) = 5A = \frac{u}{R} + C\frac{du}{dt}\Big|_{t=0} = 5 + \frac{1}{4}\left(-3A - B - 1\right) = 5 \Rightarrow 3A + B = -1$

$A = -3, B = 8$

Odakle je konačno $u(t) = -3e^{-3t} + 8e^{-t} - te^{-3t}, t \geq 0$

b) $u_g(t) = \frac{2}{3}e^{-2t}, t \geq 0$

Rješenje se traži kao zbir homogenog partikularnog:

$u(t) = u_h(t) + u_p(t)$

$u_h(t) = Ae^{s_1 t} + Be^{s_2 t}$

$u_p(t) = Ce^{-2t}$

Za određivanje $u_p(t)$ dalje nam je potrebno:

$u_p'(t) = -2Ce^{-2t}$

$u_p''(t) = 4Ce^{-2t}$

$\left(D^2 + 4D + 3\right)u_p(t) = 3 \cdot \frac{2}{3}e^{-2t}$

$4Ce^{-2t} - 8Ce^{-2t} + 3Ce^{-2t} = 2e^{-3t}$

$-Ce^{-2t} = 2e^{-2t} \Rightarrow C = -2$

Odakle je

$u_p(t) = -2e^{-2t}$

$u(t) = Ae^{-3t} + Be^{-t} - 2e^{-2t}$

Na osnovu početnih uslova je:

$u\left(0^+\right) = 5V = A + B - 2 \Rightarrow A + B = 7$

$i_L\left(0^+\right) = 5A = \frac{u}{R} + C\frac{du}{dt}\Big|_{t=0} = 5 + \frac{1}{4}\left(-3A - B + 4\right) = 5 \Rightarrow 3A + B = 4$

$A = -\frac{3}{2}, B = \frac{11}{2}$

Odakle je konačno $u(t) = -\frac{3}{2}e^{-3t} + \frac{11}{2}e^{-t} - 2e^{-2t}, t \geq 0$

c) $u_g(t) = 3t^2, t \ge 0$

Rješenje se traži kao zbir homogenog partikularnog:

$$u(t) = u_h(t) + u_p(t)$$

$$u_h(t) = Ae^{s_1 t} + Be^{s_2 t}$$

$$u_p(t) = Ct^2 + Dt + E$$

Za određivanje $u_p(t)$ dalje nam je potrebno:

$$u_p'(t) = 2Ct + D$$

$$u_p''(t) = 2C$$

$$\left(D^2 + 4D + 3\right)u_p(t) = 3t^2$$

$$2C + 8Ct + 4D + 3Ct^2 + 3Dt + 3E = 3t^2$$

$$3Ct^2 + (8C + 3D)t + (2C + 4D + 3E) = 3t^2$$

$$\Rightarrow C = 1, D = -\frac{8}{3}, E = \frac{26}{9}$$

Odakle je

$$u_p(t) = t^2 - \frac{8}{3}t + \frac{26}{9}$$

$$u(t) = Ae^{-3t} + Be^{-t} + t^2 - \frac{8}{3}t + \frac{26}{9}$$

Na osnovu početnih uslova je:

$$u\left(0^+\right) = 5V = A + B + \frac{26}{9} \Rightarrow A + B = \frac{19}{9}$$

$$i_L\left(0^+\right) = 5A = \frac{u}{R} + C\frac{du}{dt}\Big|_{t=0} = 5 + \frac{1}{4}\left(-3A - B - \frac{8}{3}\right) = 5 \Rightarrow 3A + B = -\frac{8}{3}$$

$$A = -\frac{43}{18}, B = \frac{9}{2}$$

Odakle je konačno $u(t) = -\frac{43}{18}e^{-3t} + \frac{9}{2}e^{-t} + t^2 - \frac{8}{3}t + \frac{26}{9}, t \ge 0$

d) $u_g(t) = \frac{5}{3}\cos t, t \ge 0$

Rješenje se traži kao zbir homogenog partikularnog:

$$u(t) = u_h(t) + u_p(t)$$

$$u_h(t) = Ae^{s_1 t} + Be^{s_2 t}$$

$$u_p(t) = C\cos t + D\sin t$$

Za određivanje $u_p(t)$ dalje nam je potrebno:

$$u_p'(t) = -C\sin t + D\cos t$$

$$u_p''(t) = -C\cos t - D\sin t$$

$$\left(D^2 + 4D + 3\right)u_p(t) = 3 \cdot \frac{5}{3}\cos t$$

$$-C\cos t - D\sin t - 4C\sin t + 4D\cos t + 3C\cos t + 3D\sin t = 5\cos t$$

$$(2C+4D)\cos t + (2D-4C)\sin t = 5\cos t$$

$$2C+4D = 5 \wedge 2D-4C = 0$$

$$\Rightarrow C = \frac{1}{2}, D = 1$$

Odakle je

$$u_p(t) = \frac{1}{2}\cos t + \sin t$$

$$u(t) = Ae^{-3t} + Be^{-t} + \frac{1}{2}\cos t + \sin t$$

Na osnovu početnih uslova je:

$$u(0^+) = 5V = A + B + \frac{1}{2} \Rightarrow A + B = \frac{9}{2}$$

$$i_L(0^+) = 5A = \frac{u}{R} + C\frac{du}{dt}\Big|_{t=0} = 5 + \frac{1}{4}(-3A - B + 1) = 5 \Rightarrow 3A + B = 1$$

$$A = -\frac{7}{4}, B = \frac{25}{4}$$

Odakle je konačno $u(t) = -\frac{7}{4}e^{-3t} + \frac{25}{4}e^{-t} + \frac{1}{2}\cos t + \sin t, t \geq 0$

e) $u_g(t) = \frac{2}{3}\sin 2t, t \geq 0$

Rješenje se traži kao zbir homogenog partikularnog:

$$u(t) = u_h(t) + u_p(t)$$

$$u_h(t) = Ae^{s_1 t} + Be^{s_2 t}$$

$$u_p(t) = e^t(C\cos 2t + D\sin 2t)$$

Za određivanje $u_p(t)$ dalje nam je potrebno:

$$u_p'(t) = e^t(C\cos 2t + D\sin 2t) + e^t(-2C\sin 2t + 2D\cos 2t)$$

$$u_p''(t) = e^t(C\cos 2t + D\sin 2t) + 2e^t(-2C\sin 2t + 2D\cos 2t) + e^t(-4C\cos 2t - 4D\sin 2t)$$

$$(D^2 + 4D + 3)u_p(t) = 3\cdot\frac{2}{3}e^t\sin 2t$$

$$e^t(C\cos 2t + D\sin 2t - 4C\sin 2t + 4D\cos 2t - 4C\cos 2t - 4D\sin 2t) +$$

$$+4e^t(C\cos 2t + D\sin 2t - 2C\sin 2t + 2D\cos 2t) +$$

$$+3e^t(C\cos 2t + D\sin 2t) = 2e^t\sin 2t$$

$$(4C + 12D)e^t\cos 2t + (-12C + 4D)e^t\sin 2t = 2e^t\sin 2t$$

$$4C + 12D = 0 \wedge -12C + 4D = 2$$

$$\Rightarrow C = -\frac{3}{20}, D = \frac{1}{20}$$

Odakle je

$$u_p(t) = \left(-\frac{3}{20}\cos 2t + \frac{1}{20}\sin 2t\right)e^t$$

$u(t) = Ae^{-3t} + Be^{-t} + \left(-\frac{3}{20}\cos 2t + \frac{1}{20}\sin 2t\right)e^{t}$

Na osnovu početnih uslova je:

$u(0^+) = 5V = A + B - \frac{3}{20} \Rightarrow A + B = \frac{103}{20}$

$i_L(0^+) = 5A = \frac{u}{R} + C\frac{du}{dt}\Big|_{t=0} = 5 + \frac{1}{4}\left(-3A - B - \frac{3}{20} + \frac{2}{20}\right) = 5 \Rightarrow 3A + B = -\frac{1}{20}$

$A = -2.6, B = 7.75$

Odakle je konačno $u(t) = -2.6e^{-3t} + 7.75e^{-t} + \left(-\frac{3}{20}\cos 2t + \frac{1}{20}\sin 2t\right)e^{t}, t \geq 0$

2. Neka je električno kolo opisano diferencijalnom jednačinom $\frac{d^2u}{dt^2} + a\frac{du}{dt} + bu = f(t)$. Naći vrijednost napona $u(t)$, ako je $u(0^+) = 3V$, $u'(0^+) = 2$, i ako je:

a) $a = 2,\ b = 5,\ f(t) = 4e^{-t}\sin 2t, t \geq 0$

b) $a = 0,\ b = 4,\ f(t) = 2\sin 2t, t \geq 0$

a) Karakteristična jednačina je oblika:

$s^2 + 2s + 5 = 0 \Rightarrow s_{1/2} = \frac{-2 \pm \sqrt{4-20}}{2} \Rightarrow s_{1/2} = -1 \pm j2$

Rješenje se traži kao zbir homogenog partikularnog:

$u(t) = u_h(t) + u_p(t)$

$u_h(t) = e^{-t}(A\cos 2t + B\sin 2t)$

$u_p(t) = te^{-t}(C\cos 2t + D\sin 2t)$

Za određivanje $u_p(t)$ dalje nam je potrebno:

$u_p'(t) = e^{-t}(C\cos 2t + D\sin 2t) - te^{-t}(C\cos 2t + D\sin 2t) + te^{-t}(-2C\sin 2t + 2D\cos 2t) =$

$= e^{-t}(C\cos 2t + D\sin 2t) + te^{-t}((2D - C)\cos 2t + (-2C - D)\sin 2t)$

$u_p''(t) = -e^{-t}(C\cos 2t + D\sin 2t) + e^{-t}(-2C\sin 2t + 2D\cos 2t) + e^{-t}((2D - C)\cos 2t + (-2C - D)\sin 2t) -$

$-te^{-t}((2D - C)\cos 2t + (-2C - D)\sin 2t) + te^{-t}(-2(2D - C)\sin 2t + 2(-2C - D)\cos 2t) =$

$= e^{-t}((4D - 2C)\cos 2t + (-4C - 2D)\sin 2t) + te^{-t}((4C - 3D)\sin 2t + (-3C - 4D)\cos 2t)$

$(D^2 + 2D + 5)u_p(t) = 4e^{-t}\sin 2t$

$e^{-t}(4D\cos 2t - 4C\sin 2t) + te^{-t}(0 \cdot \cos 2t + 0 \cdot \sin 2t) = 4e^{-t}\sin 2t$

$\Rightarrow C = -1, D = 0$

Odakle je

$u_p(t) = -te^{-t}\sin 2t$

$u(t) = e^{-t}(A\cos 2t + B\sin 2t) - te^{-t}\sin 2t$

Na osnovu početnih uslova je:

$u\left(0^+\right)=A=3$

$u'\left(0^+\right)=-e^{-t}\left(A\cos 2t+B\sin 2t\right)+e^{-t}\left(-2A\sin 2t+2B\cos 2t\right)-e^{-t}\sin 2t+te^{-t}\sin 2t-2te^{-t}\cos 2t\Big|_{t=0}$

$u'\left(0^+\right)=-A+2B=2\Rightarrow B=\dfrac{5}{2}$

Odakle je konačno $u\left(t\right)=\left(3\cos 2t+\dfrac{5}{2}\sin 2t\right)e^{-t}-te^{-t}\sin 2t, t\geq 0$

b) Karakteristična jednačina je oblika:

$s^2+4=0\Rightarrow s_1=s_2=\pm 2j$

Rješenje se traži kao zbir homogenog partikularnog:

$u\left(t\right)=u_h\left(t\right)+u_p\left(t\right)$

$u_h\left(t\right)=A\cos 2t+B\sin 2t$

$u_p\left(t\right)=t\left(C\cos 2t+D\sin 2t\right)$

Za određivanje $u_p(t)$ dalje nam je potrebno:

$u_p'\left(t\right)=\left(C\cos 2t+D\sin 2t\right)+t\left(-2C\sin 2t+2D\cos 2t\right)$

$u_p''\left(t\right)=2\cdot\left(-2C\sin 2t+2D\cos 2t\right)+t\left(-4C\cos 2t-4D\sin 2t\right)$

$\left(D^2+4\right)u_p\left(t\right)=2\sin 2t$

$4D\cdot\cos 2t-4C\cdot\sin 2t=2\sin 2t$

$\Rightarrow C=-\dfrac{1}{2}, D=0$

Odakle je

$u_p\left(t\right)=-\dfrac{1}{2}t\cos 2t$

$u\left(t\right)=A\cos 2t+B\sin 2t-\dfrac{1}{2}t\cos 2t$

Na osnovu početnih uslova je:

$u\left(0^+\right)=A=3$

$u'\left(0^+\right)=-2A\sin 2t+2B\cos 2t-\dfrac{1}{2}\cos 2t+t\sin 2t\Big|_{t=0}$

$u'\left(0^+\right)=2B-\dfrac{1}{2}=2\Rightarrow B=\dfrac{5}{4}$

Odakle je konačno $u\left(t\right)=3\cos 2t+\dfrac{5}{4}\sin 2t-\dfrac{1}{2}t\cos 2t, t\geq 0$